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Criteria for oscillation of noncanonical superlinear half-linear dynamic equations

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This article comes up with criteria to make sure that the solutions to superlinear, half-linear, and noncanonical dynamic equations oscillate in both advanced and delayed cases; these criteria are comparable to the Hille-type and Ohriska-type criteria for the canonical nonlinear dynamic equations; and also these results solve an open problem in many existing works in the literature on dynamic equations. To demonstrate the importance of the results, some examples have been introduced.

Keywords Oscillation behavior, Second-order, Nonlinear, Dynamic equation, Time scales

This research paper aims to study the oscillatory behavior of a specific class of second-order noncanonical superlinear half-linear dynamic equations of the form

$$\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1}z^{\Delta}(\tau)\right]^{\Delta}+q(\tau)|z(\varphi(\tau))|^{\gamma-1}z(\varphi(\tau))=0 \tag{1}$$

on an unbounded above arbitrary time scale $\mathbb{T}$, where $\tau\in[\tau_0,\infty)_{\mathbb{T}}$, $\tau_0\geq 0$, $\tau_0\in\mathbb{T}$, $\gamma\geq 1$, r and q are positive rd-continuous functions on $\mathbb{T}$, and $\varphi:\mathbb{T}\to\mathbb{T}$ is a rd-continuous nondecreasing function satisfying $\lim_{\tau\to\infty}\varphi(\tau)=\infty$.

By a solution of equation (1) we mean a nontrivial real-valued function $z\in \mathrm{C}^1_{\mathrm{rd}}[\mathrm{T}_z,\infty)_{\mathbb{T}}$, $T_z\in[\tau_0,\infty)_{\mathbb{T}}$ such that $r\left|z^{\Delta}\right|^{\gamma-1}z^{\Delta}\in \mathrm{C}^1_{\mathrm{rd}}[\mathrm{T}_z,\infty)_{\mathbb{T}}$ and z satisfies (1) on $[T_z,\infty)_{\mathbb{T}}$, where C_{rd} is the set of rd-continuous functions. In accordance with the findings of Trench[1], it is stated that Eq. (1) is considered to be in noncanonical form when

$$\int_{\tau_0}^{\infty}\frac{\Delta\omega}{r^{1/\gamma}(\omega)}<\infty. \tag{2}$$

Conversely, Eq. (1) is deemed to be in canonical form when

$$\int_{\tau_0}^{\infty}\frac{\Delta\omega}{r^{1/\gamma}(\omega)}=\infty. \tag{3}$$

A solution z of (1) is considered oscillatory if it does not become eventually positive or eventually negative. Otherwise, we refer to it as nonoscillatory. We will not take into account solutions that vanish in an identical manner in the neighborhood of infinity. A time scale $\mathbb{T}$ is any closed real subset. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(\tau)=\inf\{\upsilon\in\mathbb{T}:\upsilon>\tau\},$$

and the function $z^{\Delta}:\mathbb{T}\to\mathbb{R}$ is called the derivative of z on $\mathbb{T}$ and defined by

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$$z^{\Delta}(\tau)=\lim_{\upsilon\to\tau}\frac{z(\sigma(\tau))-z(\upsilon)}{\sigma(\tau)-\upsilon};$$

Stefan Hilger[2] introduced the theory of dynamic equations on time scales to unify analysis of continuous and discrete systems. Many applications use different time scales. Dynamic equation theory includes classical theories for differential and difference equations and instances in between. The q-difference equations, with significant consequences in quantum theory (refer to[3]), can be analyzed across several time scales. The time scales include $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^{\lambda}:\lambda\in\mathbb{N}_0 \text{ for } q>1\}$, $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=\mathbb{N}^2$, and $\mathbb{T}=\mathbb{T}_n$, where $\mathbb{T}_n$ denotes harmonic numbers. See the sources[4–6] for more information on time scale calculus.

Researchers in numerous applied disciplines have shown significant interest in the phenomenon of oscillation, primarily due to its foundations in mechanical vibrations and its extensive application in the realms of science and engineering. In order to incorporate the impact of temporal contexts on solutions, oscillation models may incorporate advanced terms or delays. Extensive research has been conducted on the subject of oscillation in delay equations, as demonstrated by the contributions of[7–18]. The existing literature concerning advanced oscillation is comparatively scant, comprising only a handful of studies that expressly investigate this subject[19–22].

A diverse array of models is utilized to investigate and comprehend the phenomenon of oscillation, which is prevalent in a vast array of practical applications. Specific models within the domain of mathematical biology have been enhanced to account for delay and/or oscillation effects through the incorporation of cross-diffusion terms. For a more comprehensive exploration of this topic, it is advisable to refer to the scholarly articles[23,24]. The current investigation is preoccupied with the scrutiny of differential equations, given their pivotal significance in comprehending and scrutinizing an extensive array of real-world phenomena. This study investigates the utilization of differential equations to analyze the turbulent flow of a polytrophic gas through a porous medium and non-Newtonian fluid theory. These disciplines possess substantial practical ramifications and necessitate an exhaustive comprehension of the mathematical principles that underpin them. Interested parties may refer to the aforementioned articles[25–34] for additional details.

The subsequent section presents the oscillation results for differential that are associated with the oscillation results for (1) on time scales. It also, provides an overview of the substantial contributions that this paper has provided. We will show that our findings not only unify some differential and difference equation oscillation results but can also be extended to determine oscillatory behavior in other cases. If $\mathbb{T}=\mathbb{R}$, then

$$\sigma(\tau)=\tau,\ z^{\Delta}(\tau)=z'(\tau),\ \int_{\alpha}^{\beta}z(\omega)\Delta\omega=\int_{\alpha}^{\beta}z(\omega)\mathrm{d}\omega,$$

and (1) transforms into the superlinear half-linear differential equation

$$\left[r(\tau)\left|z'(\tau)\right|^{\gamma-1}z'(\tau)\right]'+q(\tau)|z(\varphi(\tau))|^{\gamma-1}z(\varphi(\tau))=0. \tag{4}$$

The oscillatory characteristics of particular cases of equation (4) are examined by Fite[35] and showed that every solution of the linear differential equation

$$z''(\tau)+q(\tau)z(\tau)=0, \tag{5}$$

oscillates if

$$\int_{\tau_0}^{\infty}q(\omega)\mathrm{d}\omega=\infty. \tag{6}$$

Hille[36] improved condition (6) and proved that if

$$\liminf_{\tau\to\infty}\tau\int_{\tau}^{\infty}q(\omega)\mathrm{d}\omega>\frac{1}{4}, \tag{7}$$

then every solution of Eq. (5) oscillates. Erbe[37] extended Hille criterion (7) to the delay differential equation

$$z''(\tau)+q(\tau)z(\varphi(\tau))=0, \tag{8}$$

where $\varphi(\xi)\leq\xi$ and seen that if

$$\liminf_{\tau\to\infty}\tau\int_{\tau}^{\infty}\frac{\varphi(\omega)}{\omega}q(\omega)\mathrm{d}\omega>\frac{1}{4}, \tag{9}$$

then every solution of Eq. (8) oscillates. Ohriska[38] obtained that if

$$\limsup_{\tau\to\infty}\tau\int_{\tau}^{\infty}\frac{\varphi(\omega)}{\omega}q(\omega)\mathrm{d}\omega>1, \tag{10}$$

then every solution of Eq. (8) oscillates.

If $\mathbb{T}=\mathbb{Z}$, then

$$\sigma(\tau) = \tau + 1,\ z^{\Delta}(\tau) = \Delta z(\tau),\ \int_{\alpha}^{\beta} z(\omega)\Delta\omega = \sum_{\upsilon=\alpha}^{\beta-1} z(\omega),$$

and (1) gets the superlinear half-linear difference equation

$$\Delta\left[r(\tau)|\Delta z(\tau)|^{\gamma-1}\Delta z(\tau)\right] + q(\tau)|z(\varphi(\tau))|^{\gamma-1}z(\varphi(\tau)) = 0.$$

Thandapani et al.[39] studied the oscillation behaviour of equation

$$\Delta^2(z(\tau)) + q(\tau)z(\tau) = 0. \tag{11}$$

and it was proved that every solution of Eq. (11)oscillates if

$$\sum_{\omega=\tau_0}^{\infty} q(\omega) = \infty. \tag{12}$$

If $\mathbb{T} = \{\zeta : \zeta = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then

$$\sigma(\tau) = q\tau,\ z^{\Delta}(\tau) = \Delta_q z(\tau) = \frac{z(q\tau) - z(\tau)}{(q-1)\tau},\ \int_{\tau_0}^{\infty} z(\omega)\Delta\omega = \sum_{k=n_0}^{\infty} \tau\left(q-1\right)z(q^k),$$

where $\tau_0 = q^{n_0}$, and (1) converts the superlinear half-linear q-difference equation

$$\Delta_q\left[r(\tau)\left|\Delta_q z(\tau)\right|^{\gamma-1}\Delta_q z(\tau)\right] + q(\tau)|z(\varphi(\tau))|^{\gamma-1}z(\varphi(\tau)) = 0.$$

In relation to the dynamic equations, Karpuz[40] considered the canonical form of the linear dynamic equation

$$\left[r(\tau)z^{\Delta}(\tau)\right]^{\Delta} + q(\tau)z(\sigma(\tau)) = 0, \tag{13}$$

and obtained that if

$$\limsup_{\tau\to\infty} \frac{\mu(\tau)}{r(\tau)} < \infty,\ \int_{\tau_0}^{\infty} \frac{\Delta\omega}{r(\omega)} = \infty,$$

and

$$\liminf_{\tau\to\infty} \int_{\tau_0}^{\tau} \frac{\Delta\omega}{r(\omega)} \int_{\tau}^{\infty} q(\omega)\Delta\omega > \frac{1}{4}$$

then every solution of Eq. (13) oscillates. Erbe et al.[41] established the Hille oscillation criterion to include the dynamic

$$\left(\left(z^{\Delta}(\tau)\right)^{\gamma}\right)^{\Delta} + q(\tau)z^{\gamma}(\varphi(\tau)) = 0, \tag{14}$$

where $\gamma \geq 1$ is a quotient of odd positive integers and $\varphi(\tau) \leq \tau$ for $\tau \in \mathbb{T}$, and showed that if

$$\int_{\tau_0}^{\infty} \varphi^{\gamma}(\omega)q(\omega)\Delta\omega = \infty \tag{15}$$

and

$$\liminf_{\tau\to\infty} \tau^{\gamma} \int_{\sigma(\tau)}^{\infty} \left(\frac{\varphi(\omega)}{\sigma(\omega)}\right)^{\alpha} q(\omega)\Delta\omega > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{16}$$

where $l := \liminf_{\tau\to\infty} \dfrac{\tau}{\sigma(\tau)} > 0$, then every solution of Eq. (14) oscillates. Hassan et al.[42] considered the canonical form of the nonlinear functional dynamic equation (1) if (3) holds and one of the following holds:

$$\liminf_{\tau\to\infty} \left\{R^{\gamma}(\tau)\int_{\sigma(\tau)}^{\infty} \phi(\omega)q(\omega)\Delta\omega\right\} > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}; \tag{17}$$

$$\limsup_{\tau\to\infty} \left\{R^{\gamma}(\tau)\int_{\tau}^{\infty} \phi(\omega)q(\omega)\Delta\omega\right\} > 1, \tag{18}$$

where

$$l := \liminf_{\tau\to\infty} \frac{R(\tau)}{R(\sigma(\tau))} > 0$$

and

$$\phi(\tau) := \begin{cases} 1, & \varphi(\tau) \geq \tau, \\ \left(\dfrac{R(\varphi(\tau))}{R(\tau)}\right)^{\gamma}, & \varphi(\tau) \leq \tau, \end{cases}$$

with $R(\tau) := \int_{\tau_0}^{\tau} \dfrac{\Delta\omega}{r^{1/\gamma}(\omega)}$, then every solution of Eq. (1) oscillates. Hassan et al.[43] improved criterion (17) for the dynamic equation (1) and proved that if $l > 0$ and (3) holds, and

$$\liminf_{\tau\to\infty} \left\{ R^{\gamma}(\tau) \int_{\tau}^{\infty} \phi(\omega) q(\omega) \Delta\omega \right\} > \frac{\gamma^{\gamma}}{l^{\gamma(\gamma-1)}(\gamma+1)^{\gamma+1}};$$

then every solution of Eq. (1) oscillates. For further Hille and Ohriska criteria, see the papers[44–49].

Regarding the noncanonical form, Hassan et al.[50] found some interesting oscillation criteria, namely Hille-type and Ohriska-type criteria, for the delay linear dynamic equation

$$\left[r(\tau)z^{\Delta}(\tau)\right]^{\Delta} + q(\tau)z(\varphi(\tau)) = 0, \tag{19}$$

where $\varphi(\tau) \leq \tau$ and $\int_{\tau_0}^{\infty} \dfrac{\Delta\omega}{r(\omega)} < \infty$, which are as follows:

Theorem 1 (see[50]) *Every solution of Eq.* (19) *oscillates if any of the following conditions are satisfied*:

$$\liminf_{\tau\to\infty} \left\{ \left(\int_{\tau}^{\infty} \frac{\Delta\omega}{r(\omega)} \right) \left(\int_{T}^{\tau} q(\omega)\Delta\omega \right) \right\} > \frac{1}{4}; \tag{20}$$

$$\limsup_{\tau\to\infty} \left\{ \left(\int_{\tau}^{\infty} \frac{\Delta\omega}{r(\omega)} \right) \left(\int_{T}^{\tau} q(\omega)\Delta\omega \right) \right\} > 1, \tag{21}$$

for sufficiently large $T \in [\tau_0, \infty)_{\mathbb{T}}$.

Also, Hassan et al.[51] established, in particular, Hille-type and Ohriska-type oscillation criteria for dynamic equation (19) in a advanced case, i.e. $\varphi(\tau) \geq \tau$, as shown in the following theorem:

Theorem 2 (see[51]) *Every solution of Eq.* (19) *oscillates if any of the following conditions are satisfied*:

$$\liminf_{\tau\to\infty} \left\{ \left(\int_{\sigma(\tau)}^{\infty} \frac{\Delta\omega}{r(\omega)} \right) \left(\int_{T}^{\sigma(\tau)} \frac{\int_{\varphi(\omega)}^{\infty} \frac{\Delta\nu}{r(\nu)}}{\int_{\omega}^{\infty} \frac{\Delta\nu}{r(\nu)}} q(\omega)\Delta\omega \right) \right\} > \frac{1}{4}; \tag{22}$$

$$\limsup_{\tau\to\infty} \left\{ \left(\int_{\sigma(\tau)}^{\infty} \frac{\Delta\omega}{r(\omega)} \right) \left(\int_{T}^{\sigma(\tau)} \frac{\int_{\varphi(\omega)}^{\infty} \frac{\Delta\nu}{r(\nu)}}{\int_{\omega}^{\infty} \frac{\Delta\nu}{r(\nu)}} q(\omega)\Delta\omega \right) \right\} > 1, \tag{23}$$

for sufficiently large $T \in [\tau_0, \infty)_{\mathbb{T}}$.

It is important to point out that most of the prior findings, such as[35–49], concentrate on the canonical form, which means that condition (3) is satisfied. Therefore, the purpose of this paper is to extend the results of[50,51] and to deduce the oscillatory Hille-type and Ohriska-type criteria for the noncanonical superlinear half-linear dynamic equation (1) for the two cases $\varphi(\tau) \leq \sigma(\tau)$ and $\varphi(\tau) \geq \sigma(\tau)$. These results solved an open problem represented in many of his papers, e.g.,[26,43,44,50,51]. Please refer to the source[25,52–61] for more information.

Criteria for oscillation (1) when $\varphi(\tau) \leq \sigma(\tau)$

This section will provide evidence for the existence of additional oscillatory criteria that emulate the Hille and Ohriska types when $\varphi(\tau) \leq \sigma(\tau)$ in the noncanonical case.

Theorem 3 *Suppose that* (2) *holds. If for sufficiently large* $T \in [\tau_0, \infty)_{\mathbb{T}}$,

$$\liminf_{\tau\to\infty} \xi(\tau) \int_{T}^{\tau} \frac{\xi^{\gamma}(\sigma(\omega))}{\xi(\omega)} q(\omega)\, \Delta\omega > \frac{\gamma}{4}, \tag{24}$$

where

$$\xi(\tau) := \int_{\tau}^{\infty} \frac{\Delta\omega}{r^{1/\gamma}(\omega)},$$

then every solution of Eq. (1) *oscillates.*

Proof Let z be a nonoscillatory solution z of Eq. (1) on $[\tau_0, \infty)_{\mathbb{T}}$. Assume, without loss of generality, $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ on $[\tau_0, \infty)_{\mathbb{T}}$. With the same approach used in proving Case (a) of[50] Theorem 1] , we have

$$\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1}z^{\Delta}(\tau)\right]^{\Delta} < 0 \text{ and } z^{\Delta}(\tau) < 0,$$

eventually. Then, there exists $\tau_1 \in [\tau_0, \infty)_{\mathbb{T}}$ such that $\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1}z^{\Delta}(\tau)\right]^{\Delta} < 0$ and $z^{\Delta}(\tau) < 0$ on $[\tau_1, \infty)_{\mathbb{T}}$. Using Pötzsche chain rule (see[5] Theorem 1.90]), we get

$$\begin{aligned}\left(r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1}z^{\Delta}(\tau)\right)^{\Delta} &= -\left(r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma}\right)^{\Delta} = -\left(\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\gamma}\right)^{\Delta}\\ &= -\gamma\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\Delta}\\ &\quad\int_0^1\left[(1-h)r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right| + h\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}\mathrm{dh}\\ &=\gamma\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta}\\ &\quad\int_0^1\left[(1-h)r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right| + h\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}\mathrm{dh}\\ &\geq\gamma\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta}\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}.\end{aligned}$$

Therefore, (1) becomes

$$\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta} + \frac{q(\tau)}{\gamma\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}}|z(\varphi(\tau))|^{\gamma-1}z(\varphi(\tau)) \leq 0. \tag{25}$$

Define

$$u(\tau) := -\frac{z(\tau)}{r^{1/\gamma}(\tau)z^{\Delta}(\tau)}. \tag{26}$$

Then,

$$\begin{aligned}u^{\Delta}(\tau) &= -\frac{1}{r^{1/\gamma}(\tau)} - \left(\frac{1}{r^{1/\gamma}(\tau)z^{\Delta}(\tau)}\right)^{\Delta}z^{\sigma}(\tau)\\ &= -\frac{1}{r^{1/\gamma}(\tau)} + \frac{\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta}}{r^{1/\gamma}(\tau)z^{\Delta}(\tau)\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\sigma}}z^{\sigma}(\tau).\end{aligned}$$

Thanks to (25) and (26), we have

$$\begin{aligned}u^{\Delta}(\tau) &\leq -\frac{1}{r^{1/\gamma}(\tau)}\\ &\quad - \frac{1}{\gamma}\frac{q(\tau)z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}}\frac{1}{r^{1/\gamma}(\tau)z^{\Delta}(\tau)}\left(\frac{z(\tau)}{r^{1/\gamma}(\tau)z^{\Delta}(\tau)}\right)^{\sigma}\\ &= -\frac{1}{r^{1/\gamma}(\tau)} - \frac{1}{\gamma}\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}z(\tau)}q(\tau)u(\tau)u^{\sigma}(\tau).\end{aligned} \tag{27}$$

Since $\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma}\right]^{\Delta} > 0$, then

$$\begin{aligned}z^{\sigma}(\tau) &\geq \int_{\sigma(\tau)}^{\infty}\frac{\left[r(\omega)\left|z^{\Delta}(\omega)\right|^{\gamma}\right]^{1/\gamma}}{r^{1/\gamma}(\omega)}\Delta\omega\\ &\geq \left(\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma}\right]^{1/\gamma}\right)^{\sigma}\int_{\sigma(\tau)}^{\infty}\frac{\Delta\omega}{r^{1/\gamma}(\omega)}\\ &= \left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\xi^{\sigma}(\tau),\end{aligned} \tag{28}$$

which implies that

$$\frac{1}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}} \geq \left[\left(\frac{\xi(\tau)}{z(\tau)}\right)^{\sigma}\right]^{\gamma-1}.$$

Therefore,

$$\begin{aligned}\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}z(\tau)} &\geq \left(\xi^{\sigma}(\tau)\right)^{\gamma-1}\left(\frac{z(\varphi(\tau))}{z^{\sigma}(\tau)}\right)^{\gamma}\frac{z^{\sigma}(\tau)}{z(\tau)} \\ &\geq \left(\xi^{\sigma}(\tau)\right)^{\gamma-1}\frac{z^{\sigma}(\tau)}{z(\tau)}.\end{aligned} \tag{29}$$

From (25) , we have $\left[r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right]^{\Delta} < 0$ and then

$$-z(\tau) < \int_{\tau}^{\infty} \frac{r^{1/\gamma}(\omega)z^{\Delta}(\omega)}{r^{1/\gamma}(\omega)}\Delta\omega \leq r^{1/\gamma}(\tau)z^{\Delta}(\tau)\int_{\tau}^{\infty}\frac{\Delta\omega}{r^{1/\gamma}(\omega)} = r^{1/\gamma}(\tau)z^{\Delta}(\tau)\xi(\tau),$$

which implies that

$$\begin{aligned}\frac{z^{\sigma}(\tau)}{z(\tau)} &= 1+\mu(\tau)\frac{z^{\Delta}(\tau)}{z(\tau)} \\ &\geq 1-\frac{\mu(\tau)}{r^{1/\gamma}(\tau)\xi(\tau)} \\ &= \frac{1}{\xi(\tau)}\left[\xi(\tau)-\frac{\mu(\tau)}{r^{1/\gamma}(\tau)}\right] \\ &= \frac{\xi^{\sigma}(\tau)}{\xi(\tau)}.\end{aligned} \tag{30}$$

Hence, from (29) and (30) , we see

$$\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}z(\tau)} \geq \frac{(\xi^{\sigma}(\tau))^{\gamma}}{\xi(\tau)}. \tag{31}$$

Substituting (31) into (27) , we have

$$u^{\Delta}(\tau) \leq -\frac{1}{r^{1/\gamma}(\tau)} - \frac{1}{\gamma}\frac{(\xi^{\sigma}(\tau))^{\gamma}}{\xi(\tau)}q(\tau)u(\tau)u^{\sigma}(\tau). \tag{32}$$

By integrating (32) from τ to v, we get

$$u(v)-u(\tau) \leq -\int_{\tau}^{v}\frac{\Delta\omega}{r^{1/\gamma}(\omega)} - \frac{1}{\gamma}\int_{\tau}^{v}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)u(\omega)u^{\sigma}(\omega)\,\Delta\omega.$$

By virtue of $u > 0$, and $u^{\Delta} < 0$, and letting $v \to \infty$, we see

$$-u(\tau) \leq -\xi(\tau) - \frac{1}{\gamma}\int_{\tau}^{\infty}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)u(\omega)u^{\sigma}(\omega)\,\Delta\omega. \tag{33}$$

Let

$$Q(\tau,\tau_1) := \int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega.$$

By multiplying both sides of (33) by $Q(\tau,\tau_1)$, we obtain

$$Q(\tau,\tau_1)\xi(\tau) \leq Q(\tau,\tau_1)u(\tau) - \frac{1}{\gamma}Q(\tau,\tau_1)\int_{\tau}^{\infty}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)u(\omega)u^{\sigma}(\omega)\,\Delta\omega. \tag{34}$$

By integrating (25) from τ_1 to τ and using (31) , we achieve that

$$\begin{aligned}r^{1/\gamma}(\tau)z^{\Delta}(\tau) &\leq r^{1/\gamma}(\tau)z^{\Delta}(\tau) - r^{1/\gamma}(\tau_1)z^{\Delta}(\tau_1) \\ &\leq \frac{-1}{\gamma}\int_{\tau_1}^{\tau}\frac{z^{\gamma}(\varphi(\omega))}{\left[\left(r^{1/\gamma}(\omega)\left|z^{\Delta}(\omega)\right|\right)^{\sigma}\right]^{\gamma-1}}q(\omega)\,\Delta\omega \\ &\leq \frac{-1}{\gamma}\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)z(\omega)\,\Delta\omega \\ &\leq \frac{-1}{\gamma}z(\tau)\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega \\ &= \frac{-1}{\gamma}z(\tau)Q(\tau,\tau_1),\end{aligned}$$

which implies that

$$0 \leq \mathbb{V} := \liminf_{\tau\to\infty} Q(\tau,\tau_1)u(\tau) \leq \gamma.$$

Therefore, for any $\varepsilon \in (0,1)$, there exists $\tau_2 \in [\tau_1, \infty)_{\mathbb{T}}$ such that, for $\tau \in [\tau_2, \infty)_{\mathbb{T}}$,

$$Q(\tau,\tau_1)u(\tau) \geq \mathbb{V} - \varepsilon. \tag{35}$$

It follows from (34) and (35) that

$$\begin{aligned}\xi(\tau)Q(\tau,\tau_1) \leq \\ &\leq Q(\tau,\tau_1)u(\tau) \\ &\quad - \frac{1}{\gamma}Q(\tau,\tau_1)(\mathbb{V}-\varepsilon)^2 \int_\tau^\infty \frac{1}{Q(\omega,\tau_1)Q^\sigma(\omega,\tau_1)} \frac{(\xi^\sigma(\omega))^\gamma}{\xi(\omega)} q(\omega)\,\Delta\omega \\ &= Q(\tau,\tau_1)u(\tau) - \frac{1}{\gamma}Q(\tau,\tau_1)(\mathbb{V}-\varepsilon)^2 \int_\tau^\infty \left(\frac{-1}{Q(\omega,s_1)}\right)^\Delta \Delta\omega \\ &= Q(\tau,\tau_1)u(\tau) - \frac{1}{\gamma}(\mathbb{V}-\varepsilon)^2,\end{aligned} \tag{36}$$

since $Q(\tau,\tau_1) \to \infty$ as $\tau \to \infty$. We obtain by utilising the liminf of the inequality (36) as $\tau \to \infty$,

$$\liminf_{\tau\to\infty} \xi(\tau)Q(\tau,\tau_1) \leq \mathbb{V} - \frac{1}{\gamma}(\mathbb{V}-\varepsilon)^2.$$

Since $\varepsilon > 0$ is an arbitrary, we achieve

$$\liminf_{\tau\to\infty} \xi(\tau)Q(\tau,\tau_1) \leq \mathbb{V} - \frac{1}{\gamma}\mathbb{V}^2 \leq \frac{\gamma}{4},$$

that is in opposition to (24). $\square$

Example 1 Consider the second-order delay nonlinear dynamic equation

$$\left[\left(\tau\sigma(\tau)z^\Delta(\tau)\right)^2 \operatorname{sgn}\left(z^\Delta(\tau)\right)\right]^\Delta + \frac{\sigma^2(\tau)}{\tau} z^2(\varphi(\tau))\operatorname{sgn}(z(\varphi(\tau))) = 0 \tag{37}$$

It is easy that (2) holds since

$$\int_{\tau_0}^\infty \frac{\Delta\omega}{r^{1/\gamma}(\omega)} = \int_{\tau_0}^\infty \frac{\Delta\omega}{\omega\sigma(\omega)} = \int_{\tau_0}^\infty \left(\frac{-1}{\omega}\right)^\Delta \Delta\omega < \infty.$$

We have

$$\begin{aligned}&\liminf_{\tau\to\infty} \xi(\tau) \int_T^\tau \frac{\xi^\gamma(\sigma(\omega))}{\xi(\omega)} q(\omega)\,\Delta\omega \\ &= \liminf_{\tau\to\infty} \int_\tau^\infty \frac{\Delta\omega}{\omega\sigma(\omega)} \int_T^\tau \frac{\left(\int_{\sigma(\omega)}^\infty \frac{\Delta\nu}{\nu\sigma(\nu)}\right)^2}{\int_\omega^\infty \frac{\Delta\nu}{\nu\sigma(\nu)}} \frac{\sigma^2(\omega)}{\omega}\,\Delta\omega \\ &= \liminf_{\tau\to\infty} \int_\tau^\infty \left(\frac{-1}{\omega}\right)^\Delta \Delta\omega \int_T^\tau \frac{\left(\int_{\sigma(\omega)}^\infty \left(\frac{-1}{\nu}\right)^\Delta \Delta\nu\right)^2}{\int_\omega^\infty \left(\frac{-1}{\nu}\right)^\Delta \Delta\nu} \frac{\sigma^2(\omega)}{\omega}\,\Delta\omega \\ &= 1.\end{aligned}$$

According to an application of Theorem 3, we have every solution of Eq. (37) oscillates.

Theorem 4 *Suppose that (2) holds. If for sufficiently large $T \in [\tau_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{\tau\to\infty} \xi(\tau) \int_T^\tau \frac{\xi^\gamma(\sigma(\omega))}{\xi(\omega)} q(\omega)\,\Delta\omega > \gamma, \tag{38}$$

then every solution of Eq. (1) oscillates.

Proof Let z be a nonoscillatory solution z of Eq. (1) on $[\tau_0, \infty)_{\mathbb{T}}$. Assume, without loss of generality, $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ on $[\tau_0, \infty)_{\mathbb{T}}$. As demonstrated in the proof of Theorem 3, there exists $\tau_1 \in [\tau_0, \infty)_{\mathbb{T}}$ such that $\left[r(\tau)\left|z^\Delta(\tau)\right|^{\gamma-1} z^\Delta(\tau)\right]^\Delta < 0$ and $z^\Delta(\tau) < 0$ on $[\tau_1, \infty)_{\mathbb{T}}$, and for $\tau \in [\tau_1, \infty)_{\mathbb{T}}$,

$$\left(r^{1/\gamma}(\tau)z^\Delta(\tau)\right)^\Delta + \frac{1}{\gamma}q(\tau)\frac{z^\gamma(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^\Delta(\tau)\right|\right)^\sigma\right]^{\gamma-1}} \leq 0, \tag{39}$$

and from (31) , we have for $\tau \in [\tau_1, \infty)_{\mathbb{T}}$,

$$\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}} \geq \frac{(\xi^{\sigma}(\tau))^{\gamma}}{\xi(\tau)}z(\tau).$$

Therefore,

$$\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta} + \frac{1}{\gamma}\frac{(\xi^{\sigma}(\tau))^{\gamma}}{\xi(\tau)}q(\tau)z(\tau) \leq 0. \tag{40}$$

By integrating (40) from τ_1 to τ, we obtain

$$\begin{aligned} r^{1/\gamma}(\tau)z^{\Delta}(\tau) \leq & r^{1/\gamma}(\tau)z^{\Delta}(\tau) - r^{1/\gamma}(\tau_1)z^{\Delta}(\tau_1) \\ & \leq -\frac{1}{\gamma}\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)z(\omega)\,\Delta\omega \\ & \leq -\frac{1}{\gamma}z(\tau)\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega, \end{aligned}$$

which implies for $\omega \in [\tau, \infty)$ and $\tau \in [\tau_1, \infty)$, that

$$r^{1/\gamma}(\omega)z^{\Delta}(\omega) \leq r^{1/\gamma}(\tau)z^{\Delta}(\tau) \leq -\frac{1}{\gamma}z(\tau)\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega. \tag{41}$$

We have for $\nu \in [\tau, \infty)$,

$$-z(\tau) \leq -z(\tau) + z(\nu) = \int_{\tau}^{\nu}\frac{r^{1/\gamma}(\omega)z^{\Delta}(\omega)}{r^{1/\gamma}(\omega)}\,\Delta\omega.$$

Letting $\nu \to \infty$, we obtain

$$-z(\tau) \leq \int_{\tau}^{\infty}\frac{r^{1/\gamma}(\omega)z^{\Delta}(\omega)}{r^{1/\gamma}(\omega)}\,\Delta\omega. \tag{42}$$

Substituting (41) into (42) , we get

$$-z(\tau) \leq -\frac{1}{\gamma}z(\tau)\left(\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega\right)\left(\int_{\tau}^{\infty}\frac{\Delta\omega}{r(\omega)}\right),$$

so

$$\xi(\tau)\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega \leq \gamma,$$

which implies that

$$\limsup_{\tau\to\infty}\xi(\tau)\int_{\tau_1}^{\tau}\frac{(\xi^{\sigma}(\omega))^{\gamma}}{\xi(\omega)}q(\omega)\,\Delta\omega \leq \gamma,$$

that is in opposition to (38). □

Example 2 Consider the second-order nonlinear delay dynamic equation

$$\left[\left(\tau^{2}\sigma(\tau)z^{\Delta}(\tau)\right)^{3}\right]^{\Delta} + 32\beta\xi(\tau)\sigma^{7}(\tau)z^{3}(\varphi(\tau)) = 0, \tag{43}$$

where $\beta > 0$ is a constant. It is easy that (2) holds due to

$$\int_{\tau_0}^{\infty}\frac{\Delta\omega}{\omega^{2}\sigma(\omega)} < \infty,$$

over such time scales $[\tau_0, \infty)_{\mathbb{T}}$, when $\int_{\tau_0}^{\infty}\frac{\Delta\omega}{\omega^{p}} < \infty$ with $p > 1$. Now

$$\begin{aligned} &\limsup_{\tau\to\infty}\xi(\tau)\int_{T}^{\tau}\frac{\xi^{\gamma}(\sigma(\omega))}{\xi(\omega)}q(\omega)\,\Delta\omega \\ &= 32\beta\limsup_{\tau\to\infty}\int_{\tau}^{\infty}\frac{\Delta\omega}{\omega^{2}\sigma(\omega)}\int_{T}^{\tau}\sigma^{7}(\omega)\left(\int_{\sigma(\omega)}^{\infty}\frac{\Delta\nu}{\nu^{2}\sigma(\nu)}\right)^{3}\Delta\omega. \end{aligned} \tag{44}$$

Since, by Pötzsche chain rule, we have

$$\left(v^2\right)^{\Delta} = 2\int_0^1 [(1-h)v + h\sigma(v)]\mathrm{d}h \le 2\sigma(\mathrm{v}),$$

and so

$$\left(\frac{-1}{v^2}\right)^{\Delta} = \frac{\left(v^2\right)^{\Delta}}{v^2\sigma^2(v)} \le \frac{2}{v^2\sigma(v)}. \tag{45}$$

From (44) and (45) , we get

$$\begin{aligned}
&\limsup_{\tau\to\infty} \xi(\tau)\int_T^\tau \frac{\xi^{\gamma}(\sigma(\omega))}{\xi(\omega)} q(\omega)\,\Delta\omega \\
&\ge 2\beta \limsup_{\tau\to\infty} \int_\tau^\infty \left(\frac{-1}{\omega^2}\right)^{\Delta}\Delta\omega \int_T^\tau \sigma^7(\omega)\left(\int_{\sigma(\omega)}^\infty \left(\frac{-1}{v^2}\right)^{\Delta}\Delta v\right)^3 \Delta\omega \\
&\ge \beta \limsup_{\tau\to\infty} \frac{1}{\tau^2}\int_T^\tau \left(\omega^2\right)^{\Delta}\Delta\omega = \beta.
\end{aligned}$$

By using Theorem 4, then every solution of Eq. (43) oscillates if $\beta > 3$ and over such time scale when $\int_{\tau_0}^\infty \frac{\Delta\omega}{\omega^p} < \infty$ with $p > 1$.

Criteria for oscillation (1) when $\varphi(\tau) \ge \sigma(\tau)$

In the following, we shall apply the oscillation criteria that were established in the preceding section to the case of $\varphi(\tau) \ge \sigma(\tau)$.

Theorem 5 *Suppose that* (2) *holds. If for sufficiently large* $T \in [\tau_0,\infty)_{\mathbb{T}}$,

$$\liminf_{\tau\to\infty} \xi(\tau)\int_T^\tau \frac{\xi^{\gamma}(\varphi(\omega))}{\xi(\omega)} q(\omega)\,\Delta\omega > \frac{\gamma}{4}, \tag{46}$$

then every solution of Eq. (1) *oscillates.*

Proof Let z be a nonoscillatory solution z of Eq. (1) on $[\tau_0,\infty)_{\mathbb{T}}$. Assume, without loss of generality, $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ on $[\tau_0,\infty)_{\mathbb{T}}$. As shown in the proof of Theorem 3, there exists $\tau_1 \in [\tau_0,\infty)_{\mathbb{T}}$ such that $\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1}z^{\Delta}(\tau)\right]^{\Delta} < 0$ and $z^{\Delta}(\tau) < 0$ on $[\tau_1,\infty)_{\mathbb{T}}$, and for $\tau \in [\tau_1,\infty)_{\mathbb{T}}$,

$$u^{\Delta}(\tau) \le -\frac{1}{r^{1/\gamma}(\tau)} - \frac{1}{\gamma}\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}z(\tau)} q(\tau)u(\tau)u^{\sigma}(\tau), \tag{47}$$

where u is defined by (26) . By using the fact that $\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma}\right]^{\Delta} > 0$, we obtain

$$\begin{aligned}
z(\varphi(\tau)) &\ge \int_{\varphi(\tau)}^\infty \frac{\left[r(\omega)\left|z^{\Delta}(\omega)\right|^{\gamma}\right]^{1/\gamma}}{r^{1/\gamma}(\omega)}\Delta\omega \\
&\ge \left[r(\varphi(\tau))\left|z^{\Delta}(\varphi(\tau))\right|^{\gamma}\right]^{1/\gamma}\int_{\varphi(\tau)}^\infty \frac{\Delta\omega}{r^{1/\gamma}(\omega)} \\
&\ge \left(\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma}\right]^{1/\gamma}\right)^{\sigma}\int_{\varphi(\tau)}^\infty \frac{\Delta\omega}{r^{1/\gamma}(\omega)} \\
&= \left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\xi(\varphi(\tau)),
\end{aligned}$$

which implies that

$$\frac{1}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}} \ge \left[\frac{\xi(\varphi(\tau))}{z(\varphi(\tau))}\right]^{\gamma-1}.$$

Therefore,

$$\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}z(\tau)} \ge \xi^{\gamma-1}(\varphi(\tau))\frac{z(\varphi(\tau))}{z(\tau)}. \tag{48}$$

By using $\left[r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right]^{\Delta} < 0$, we have

$$-z(\tau) < \int_\tau^\infty \frac{r^{1/\gamma}(\omega)z^{\Delta}(\omega)}{r^{1/\gamma}(\omega)}\Delta\omega \le r^{1/\gamma}(\tau)z^{\Delta}(\tau)\int_\tau^\infty \frac{\Delta\omega}{r^{1/\gamma}(\omega)} = r^{1/\gamma}(\tau)z^{\Delta}(\tau)\xi(\tau),$$

which yields that

$$\left(\frac{z(\tau)}{\xi(\tau)}\right)^{\Delta} = \frac{\xi(\tau)z^{\Delta}(\tau) - \xi^{\Delta}(\tau)z(\tau)}{\xi(\tau)\xi^{\sigma}(\tau)} = \frac{r^{1/\gamma}(\tau)\xi(\tau)z^{\Delta}(\tau) + z(\tau)}{r^{1/\gamma}(\tau)\xi(\tau)\xi^{\sigma}(\tau)} > 0$$

Hence,

$$\frac{z(\varphi(\tau))}{z(\tau)} \geq \frac{\xi(\varphi(\tau))}{\xi(\tau)}. \tag{49}$$

Substituting (49) into (48) , we get

$$\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1} z(\tau)} \geq \frac{\xi^{\gamma}(\varphi(\tau))}{\xi(\tau)}. \tag{50}$$

From (47) and (50) , we obtain for $\tau \in [\tau_1, \infty)_{\mathbb{T}}$,

$$u^{\Delta}(\tau) \leq -\frac{1}{r^{1/\gamma}(\tau)} - \frac{1}{\gamma}\frac{\xi^{\gamma}(\varphi(\tau))}{\xi(\tau)} q(\tau)u(\tau)u^{\sigma}(\tau).$$

The rest of the evidence is the same as it is in the proof of Theorem 3, hence is omitted. □

Example 3 Consider the second-order nonlinear advanced dynamic equation

$$\left[\left(\tau^2 z^{\Delta}(\tau)\right)^2 \mathrm{sgn}\left(z^{\Delta}(\tau)\right)\right]^{\Delta} + \xi(\tau)(\varphi(\tau)z(\varphi(\tau)))^2 \ \mathrm{sgn}(z(\varphi(\tau))) = 0 \tag{51}$$

Therefore,

$$\begin{aligned}
&\liminf_{\tau\to\infty} \xi(\tau) \int_T^{\tau} \frac{\xi^{\gamma}(\varphi(\omega))}{\xi(\omega)} q(\omega)\Delta\omega \\
&= \liminf_{\tau\to\infty} \int_{\tau}^{\infty} \frac{\Delta\omega}{\omega^2} \int_T^{\tau} \left(\varphi(\omega)\int_{\varphi(\omega)}^{\infty} \frac{\Delta\nu}{\nu^2}\right)^2 \Delta\omega \\
&\geq \liminf_{\tau\to\infty} \int_{\tau}^{\infty} \left(\frac{-1}{\omega}\right)^{\Delta} \Delta\omega \int_T^{\tau} \left(\varphi(\omega)\int_{\varphi(\omega)}^{\infty} \left(\frac{-1}{\nu}\right)^{\Delta} \Delta\nu\right)^2 \Delta\omega \\
&= 1 > \frac{\gamma}{4}.
\end{aligned}$$

Using Theorem 5, then every solution of Eq. (51) oscillates over such time scale when $\int_{\tau_0}^{\infty} \frac{\Delta\omega}{\omega^p} < \infty$ with $p > 1$.

Theorem 6 *Suppose that* (2) *holds. If for sufficiently large* $T \in [\tau_0, \infty)_{\mathbb{T}}$,

$$\limsup_{\tau\to\infty} \xi(\tau) \int_T^{\tau} \frac{\xi^{\gamma}(\varphi(\omega))}{\xi(\omega)} q(\omega)\ \Delta\omega > \gamma, \tag{52}$$

then every solution of Eq. (1) *oscillates.*

Proof Let z be a nonoscillatory solution z of Eq. (1) on $[\tau_0, \infty)_{\mathbb{T}}$. Assume, without loss of generality, $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ on $[\tau_0, \infty)_{\mathbb{T}}$. As explained in the proof of[50] Theorem 1], we have

$$\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1} z^{\Delta}(\tau)\right]^{\Delta} < 0 \text{ and } z^{\Delta}(\tau) < 0,$$

eventually. there exists $\tau_1 \in [\tau_0, \infty)_{\mathbb{T}}$ such that $\left[r(\tau)\left|z^{\Delta}(\tau)\right|^{\gamma-1} z^{\Delta}(\tau)\right]^{\Delta} < 0$ and $z^{\Delta}(\tau) < 0$ on $[\tau_1, \infty)_{\mathbb{T}}$. As demonstrated in the proof of Theorem 3, we see for $\tau \in [\tau_1, \infty)_{\mathbb{T}}$,

$$\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta} + \frac{1}{\gamma} q(\tau) \frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}} \leq 0,$$

and from the proof of Theorem 5,

$$\frac{z^{\gamma}(\varphi(\tau))}{\left[\left(r^{1/\gamma}(\tau)\left|z^{\Delta}(\tau)\right|\right)^{\sigma}\right]^{\gamma-1}} \geq \frac{\xi^{\gamma}(\varphi(\tau))}{\xi(\tau)} z(\tau).$$

Hence,

$$\left(r^{1/\gamma}(\tau)z^{\Delta}(\tau)\right)^{\Delta}+\frac{1}{\gamma}\frac{\xi^{\gamma}(\varphi(\tau))}{\xi(\tau)}q(\tau)z(\tau)\leq 0.$$

The rest of the evidence is the same as it is in the proof of Theorem 4, hence is omitted. □

Example 4 Consider the second-order advanced nonlinear dynamic equation

$$\left[\left(\tau\sigma(\tau)z^{\Delta}(\tau)\right)^{3}\right]^{\Delta}+\frac{3\beta}{\tau}(\varphi(\tau)z(\varphi(\tau)))^{3}=0, \tag{53}$$

where $\beta>0$ is a constant. Thus,

$$\begin{aligned}
&\limsup_{\tau\to\infty}\xi(\tau)\int_{T}^{\tau}\frac{\xi^{\gamma}(\varphi(\omega))}{\xi(\omega)}q(\omega)\,\Delta\omega\\
&=3\beta\limsup_{\tau\to\infty}\int_{\tau}^{\infty}\frac{\Delta\omega}{\omega\sigma(\omega)}\int_{T}^{\tau}\frac{\left(\varphi(\omega)\int_{\varphi(\omega)}^{\infty}\frac{\Delta\nu}{\nu\sigma(\nu)}\right)^{3}}{\omega\int_{\omega}^{\infty}\frac{\Delta\nu}{\nu\sigma(\nu)}}\,\Delta\omega\\
&=3\beta\limsup_{\tau\to\infty}\int_{\tau}^{\infty}\left(\frac{-1}{\omega}\right)^{\Delta}\Delta\omega\int_{T}^{\tau}\frac{\left(\varphi(\omega)\int_{\varphi(\omega)}^{\infty}\left(\frac{-1}{\nu}\right)^{\Delta}\Delta\nu\right)^{3}}{\omega\int_{\omega}^{\infty}\left(\frac{-1}{\nu}\right)^{\Delta}\Delta\nu}\,\Delta\omega\\
&=3\beta.
\end{aligned}$$

By application of Theorem 6, if $\beta>1$, then every solution of Eq. (53) oscillates.

Conclusions and discussion

(I) The results of this paper are applicable to all time scales, including $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, and so forth (see[6]).

(II) In contrast to previous literature, the results we have obtained in this work do not presume the fulfillment of condition (3) (canonical case), thereby resolving an open problem that has been referenced in numerous papers, as indicated in[26,43,44,50,51].

(III) This research paper introduces criteria for Hille-type and Ohriska-type oscillation that can be applied to (1) in both cases, $\varphi(\tau)\leq\sigma(\tau)$ and $\varphi(\tau)\geq\sigma(\tau)$ and on any arbitrary time scale. Also, our results extend relevant contributions to second-order dynamic equations of[50,51].

(IV) It would be interesting to find such criteria for noncanonical sublinear half-linear dynamic equations (1), where $0<\gamma\leq 1$ is a constant.

Data availability

All data generated or analysed during this study are included in this published article.

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