CHAPTER

10 Main results on U-Bernoulli–Korobov-type polynomials and their approximate roots

Clemente Cesarano[a], **Alejandro Urieles**[b], **and William Ramírez**[a,c]

[a] *Section of Mathematics Luciano Modica, International Telematic University, Uninettuno, Rome, Italy*

[b] *Programa de Matemáticas, Universidad del Atlántico, Barranquilla, Colombia*

[c] *Department of Natural and Exact Sciences, Universidad de la Costa, Barranquilla, Colombia*

10.1 Introduction and background

The study of discrete Appell polynomials holds great importance in mathematics due to their distinctive properties and broad spectrum of applications. Similar to their continuous counterparts, these polynomials are characterized by a discrete shift operator acting as their fundamental differential operator. They play a key role in the development of special functions, which have practical uses in various areas such as approximation theory, numerical analysis, quantum mechanics, and other fields in mathematics, physics, engineering, and statistics (see [9,11]).

In this context, let $f : \mathbb{Z} \to \mathbb{R}$ be any function of the integers, and consider the discrete operator $\Delta f(x) = f(x+1) - f(x)$. This operator plays a crucial role in the definition and analysis of discrete Appell polynomials, further highlighting their importance in both theoretical and applied mathematics.

A discrete Appell sequence $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials such that (see [5]):

$$\Delta p_k(x) = p_k(x+1) - p_k(x) = k p_{k-1}(x), \quad k \geq 1.$$

It is known that a Taylor-series expansion can define Appell sequences (see [1]):

$$A(z)e^{xz} = \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!}, \tag{10.1}$$

where $A(z)$ is an analytic function at $z = 0$ with $A(0) \neq 0$. Similarly, discrete Appell sequences can be defined by the Taylor-generating expansion:

$$A(z)(1+z)^x = \sum_{n=0}^{\infty} p_n(x) \frac{z^n}{n!}, \tag{10.2}$$

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where again $A(0) \neq 0$.

Typical examples include the elementary case $\{x^k\}_{k=0}^{\infty}$, whose generating function corresponds to (10.1) with $A(z) = 1$, as well as the classical Bernoulli polynomials, employed by Euler in 1740 to evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$. For the Bernoulli polynomials, the generating function (10.1) becomes $A(z) = \frac{z}{e^z - 1}$.

The discrete analogs of the Bernoulli polynomials are known as the Bernoulli polynomials of the second kind (see [3]), denoted by $b_k(x)$. Introduced independently by Jordan [10] and Rey Pastor [15] in 1929, they are also referred to as Rey–Pastor polynomials (see [2]). Their generating function can be written as:

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{k=0}^{\infty} b_k(x) \frac{z^k}{k!}.$$

Throughout this chapter, the following notations are used: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0$ the set of all non-negative integers, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}$ the set of all complex numbers.

The Korobov polynomials $K_n(x; \lambda)$ of the first kind are defined by the generating function (cf. [12]):

$$\frac{\lambda z}{(1+z)^{\lambda} - 1}(1+z)^x = \sum_{n=0}^{\infty} K_n(x; \lambda) \frac{z^n}{n!}.$$

When $x = 0$, the numbers $K_n(\lambda) = K_n(0; \lambda)$ are called the Korobov numbers.

In [4], Carlitz introduced the degenerate Bernoulli polynomials, given by the generating function:

$$\frac{z}{(1+\lambda z)^{\frac{1}{\lambda}} - 1}(1+\lambda z)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathscr{B}_n(x; \lambda) \frac{z^n}{n!}. \tag{10.3}$$

From (10.3), it follows that

$$\lim_{\lambda \to 0} \mathscr{B}_n(x; \lambda) = B_n(x), \quad (n \geq 0),$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Additionally, for $n \in \mathbb{N}_0$, we define a new family called the U-Bernoulli polynomials $M_n(x)$ of degree n by the power-series expansion at 0 of the following generating function (see [13]):

$$f(x; z) = \frac{z}{e^{-z} - 1} e^{-xz} = \sum_{n=0}^{\infty} M_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10.4}$$

We have, for the first few U-Bernoulli polynomials $M_n(x)$:

$$M_0(x) = -1, \quad M_1(x) = x - \tfrac{1}{2}, \quad M_2(x) = -x^2 + x - \tfrac{1}{6},$$
$$M_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x, \quad M_4(x) = -x^4 + 2x^3 - x^2 + \tfrac{1}{30}, \quad M_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

When $x = 0$ in (10.4), the U-Bernoulli numbers M_n are defined by the generating function:

$$f(z) = \frac{z}{e^{-z} - 1} = \sum_{n=0}^{\infty} M_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

Some of the first U-Bernoulli numbers are:

$$M_0 = -1, \quad M_1 = -\frac{1}{2}, \quad M_2 = -\frac{1}{6}, \quad M_3 = 0, \quad M_4 = \frac{1}{30}, \quad M_5 = 0.$$

On the topic of polynomial families and their various extensions, a remarkably large amount of research has appeared in the literature (see, for example, [6–8,17,18]).

Given this context, the main objective of this work is to define and study discrete U-Bernoulli–Korobov polynomials. We explore their algebraic and differential properties, and present graphical representations of their zeros, computed using a Python program.

10.2 Main results on U-Bernoulli–Korobov discrete polynomials

This section explores the properties of U-Bernoulli–Korobov discrete polynomials. We include schematic proofs to highlight the main methods and results; further details can be found in [14,16].

Definition 10.2.1. The new family of U-Bernoulli–Korobov discrete polynomials $\mathscr{P}_n(x)$ of degree n in x are defined by the generating function:

$$\left(\frac{z}{e^{-z} - 1}\right)(1 + z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10.5}$$

The first six U-Bernoulli–Korobov discrete polynomials $\mathscr{P}_n(x)$, are:

$$\mathscr{P}_0(x) = -1, \quad \mathscr{P}_1(x) = -x - \frac{1}{2}, \quad \mathscr{P}_2(x) = -x^2 - \frac{1}{6},$$
$$\mathscr{P}_3(x) = -x^3 + \frac{3}{2}x^2 - x, \quad \mathscr{P}_4(x) = -x^4 + 4x^3 - 4x^2 + 3x + \frac{1}{30},$$
$$\mathscr{P}_5(x) = -x^5 + \frac{15}{2}x^4 - \frac{65}{3}x^3 + \frac{55}{2}x^2 - \frac{33}{6}x.$$

For $x = 0$ in (10.5) the U-Bernoulli–Korobov discrete numbers $\mathscr{P}_n(0)$ are defined by the generating function:

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10.6}$$

Some of these numbers are:

$$\mathscr{P}_0 = -1; \quad \mathscr{P}_1 = -\frac{1}{2}; \quad \mathscr{P}_2 = -\frac{1}{6}; \quad \mathscr{P}_3 = 0; \quad \mathscr{P}_4 = \frac{1}{30}; \quad \mathscr{P}_5 = 0.$$

A consequence of (10.5) and (10.6) is the following proposition.

Proposition 10.2.1. *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of U-Bernoulli–Korobov discrete polynomials in the variable x. Then, the following statement holds:*

$$\mathscr{P}_n(x) = \sum_{k=0}^{\infty} \binom{x}{k} \frac{n!}{(n-k)!} \mathscr{P}_{n-k}.$$

Proof. It is sufficient to use the generatriz function given in Definition 10.2.1:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} &= \left(\frac{z}{e^{-z}-1}\right)(1+z)^x \\ &= \sum_{n=0}^{\infty} \mathscr{P}_n \frac{z^n}{n!} \sum_{m=0}^{\infty} \binom{x}{m} z^m \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{x}{k} \frac{\mathscr{P}_n - k}{(n-k)!} z^n.\end{aligned}$$

Comparing the coefficients of z^n on both sides of the equation, we have

$$\begin{aligned}\frac{\mathscr{P}_n(x)}{n!} &= \sum_{k=0}^{n} \binom{x}{k} \frac{\mathscr{P}_n - k}{(n-k)!} \\ \mathscr{P}_n(x) &= \sum_{k=0}^{n} \binom{x}{k} \frac{n!}{(n-k)!} \mathscr{P}_{n-k}.\end{aligned}$$

This completes the proof.

Theorem 10.2.1. *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of U-Bernoulli–Korobov discrete polynomials in the variable x that satisfy the following relation:*

$$\mathscr{P}_n(x) = \mathscr{P}_n(x+1) - n\mathscr{P}_{n-1}(x).$$

Proof. Of the generating function (10.5), we have:

$$\left(\frac{z}{e^z - 1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!}. \tag{10.7}$$

Multiplying by $(1+z)$ both sides of (10.7), we have:

$$\begin{aligned}
\left(\frac{z}{e^{-z} - 1}\right)(1+z)^{(x+1)} &= (1+z)\sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} \\
\sum_{n=0}^{\infty} \mathscr{P}_n(x+1)\frac{z^n}{n!} &= \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} + z\sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} + \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} + \sum_{n=1}^{\infty} \mathscr{P}_{n-1}(x)\frac{z^n}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \mathscr{P}_n(x)\frac{z^n}{n!} + \sum_{n=0}^{\infty} n\,\mathscr{P}_{n-1}(x)\frac{z^n}{n!} \\
\sum_{n=0}^{\infty} \mathscr{P}_n(x+1)\frac{z^n}{n!} &= \sum_{n=0}^{\infty} [\mathscr{P}_n(x) + n\mathscr{P}_{n-1}(x)]\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x) = \mathscr{P}_n(x+1) - n\,\mathscr{P}_{n-1}(x).$$

Theorem 10.2.2 (Differential expressions). *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of U-Bernoulli–Korobov discrete polynomials in the variable x, which satisfy the following relations:*

1.

$$(n-1)\mathscr{P}_n(x) - n\psi(x;n;z)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x) = 0,$$

where

$$\psi(x;n;z) = \left[\frac{x}{(z+1)\log(z+1)} + \frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right].$$

2.

$$\frac{\partial \mathscr{P}_n(x)}{\partial x} = \sum_{k=0}^{n-1} n\binom{n-1}{k}(-1)^k \frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

Proof. For the proof of 1. Consider the following equations:

$$L(x;z)=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}, \tag{10.8}$$

$$L(x;z)=\frac{z}{e^{-z}-1}(1+z)^x. \tag{10.9}$$

Partially differentiating with respect to z in (10.8) and (10.9), the result is:

$$\frac{\partial L(x;z)}{\partial z}=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^{n-1}}{n!}$$

and

$$\frac{\partial L(x;z)}{\partial z}=\frac{(1+z)^x}{e^{-z}-1}+\left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{x}{1+z}+\left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{e^{-z}-1}. \tag{10.10}$$

Partially differentiating with respect to x in (10.9), we have:

$$\frac{\partial L(x;z)}{\partial x}=\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}.$$

Of (10.10), we have

$$\begin{aligned}
0&=\frac{\partial L(x;z)}{\partial z}-\frac{(1+z)^x}{e^{-z}-1}-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{x}{(1+z)\log(z+1)}\\
&\quad-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\\
0&=\frac{\partial L(x;z)}{\partial z}-\frac{(1+z)^x}{e^{-z}-1}-\frac{x}{(1+z)\log(z+1)}\frac{\partial L(x;z)}{\partial x}\\
&\quad-\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\frac{\partial L(x;z)}{\partial x}\\
0&=\frac{z\partial L(x;z)}{\partial z}-\left[\frac{zx}{(1+z)\log(z+1)}+\frac{ze^{-z}}{(e^{-z}-1)\log(z+1)}\right]\frac{\partial L(x;z)}{\partial x}-\frac{z(1+z)^x}{e^{-z}-1}\\
0&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^{n-1}}{n!}z-\sum_{n=0}^{\infty}\left[\frac{zx}{(1+z)\log(z+1)}+\frac{ze^{-z}}{(e^{-z}-1)\log(z+1)}\right]\\
&\quad\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\\
0&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^n}{n!}-\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]
\end{aligned}$$

$$\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x)\frac{nz^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$0=(n-1)\mathscr{P}_n(x)-\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]n\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x).$$

This completes the proof.

Proof. For the proof of (10.1). Partially differentiating with respect to x in 10.2.1, we have:

$$\left(\frac{z}{e^{-z}-1}\right)\frac{\partial}{\partial x}\left[(1+z)^x\right]=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}\right)=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathscr{P}_{n-1-k}(x)(-1)^k\binom{n-1}{k}\frac{k!}{(k+1)}n\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\frac{\partial}{\partial x}\mathscr{P}_n(x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}(-1)^k\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

Proposition 10.2.2. *The U-Bernoulli–Korobov discrete polynomials in the variable x satisfy the following relations:*

(i) $\mathscr{P}_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}(y)_k\mathscr{P}_{n-k}(x).$

(ii) $\mathscr{P}_n(x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}(x)_k+\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_k(x).$

Proof. For the proof of (i), it is sufficient to use the generatriz function given in (10.5):

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_n(x+y)\frac{z^n}{n!}&=\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x+y}\\&=\left(\frac{z}{e^{-z}-1}\right)(1+z)^x(1+z)^y\\&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\sum_{k=0}^{n}\binom{y}{k}z^n\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (y)_k \mathscr{P}_{n-k}(x) \frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} (y)_k \mathscr{P}_{n-k}(x).$$

To prove (ii). We will examine the generating function (10.5) as follows:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!}$$

$$e^z z(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!} - e^z \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!}$$

$$z \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \binom{x}{k} z^k = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \mathscr{P}_k(x) \frac{z^k}{k!}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} (x)_k n \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left[\mathscr{P}_n(x) - \sum_{k=0}^{n} \binom{n}{k} \mathscr{P}_k(x) \right] \frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, the result is:

$$\mathscr{P}_n(x) = \sum_{k=0}^{n-1} n \binom{n-1}{k} (x)_k + \sum_{k=0}^{n} \binom{n}{k} \mathscr{P}_k(x).$$

Theorem 10.2.3. *For $n \geq 0$, let $\{\mathscr{P}_n(x)\}_{n \geq 0}$ be the sequences of U-Bernoulli–Korobov discrete polynomials in the variable x that satisfy the following relation:*

$$\sum_{k=0}^{n} \binom{n}{k} \left[\mathscr{P}_k(x+y) \mathscr{P}_{n-k} - \mathscr{P}_{n-k}(x)\, \mathscr{P}_k(y) \right] = 0.$$

Proof. Let's consider the following expressions:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!} \tag{10.11}$$

and

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^y = \sum_{n=0}^{\infty} \mathscr{P}_n(y) \frac{z^n}{n!}. \tag{10.12}$$

Of (10.11) and (10.12), we have

$$\left[\frac{z}{e^{-z}-1}\right]^2(1+z)^{x+y}=\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$

$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x+y)\frac{z^n}{n!}\right)=\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y)\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y)\frac{z^n}{n!}$$

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y).$$

Therefore

$$\sum_{k=0}^{n}\binom{n}{k}[\mathscr{P}_k(x+y)\mathscr{P}_{n-k}-\mathscr{P}_{n-k}(x)\,\mathscr{P}_k(y)]=0.$$

Theorem 10.2.4. *For $n\in\mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of U-Bernoulli–Korobov discrete polynomials in the variable x that satisfy the following relation:*

$$\mathscr{P}_n(x)=\mathscr{P}_n+\sum_{k=0}^{n-1}\frac{n}{(k+1)}\binom{n-1}{k}(x)_{k+1}\mathscr{P}_{n-1-k}.$$

Proof. Of (10.5) and (10.6), we obtain:

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$

and

$$\left(\frac{z}{e^{-z}-1}\right)=\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!},$$

then

$$\begin{aligned}\sum_{n=0}^{\infty}[\mathscr{P}_n(x)-\mathscr{P}_n]\frac{z^n}{n!}&=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}-\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\\&=\frac{z}{e^{-}z-1}\left[(1+z^x)-1\right]\\&=\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\sum_{n=0}^{\infty}\binom{x}{n+1}z^{n+1}\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k+1}\frac{\mathscr{P}_{n-k}}{(n-k)!}z^{n+1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\frac{n}{(k+1)}\binom{n-1}{k}(x)_{k+1}\mathscr{P}_{n-k-1}\frac{z^n}{n!}.$$

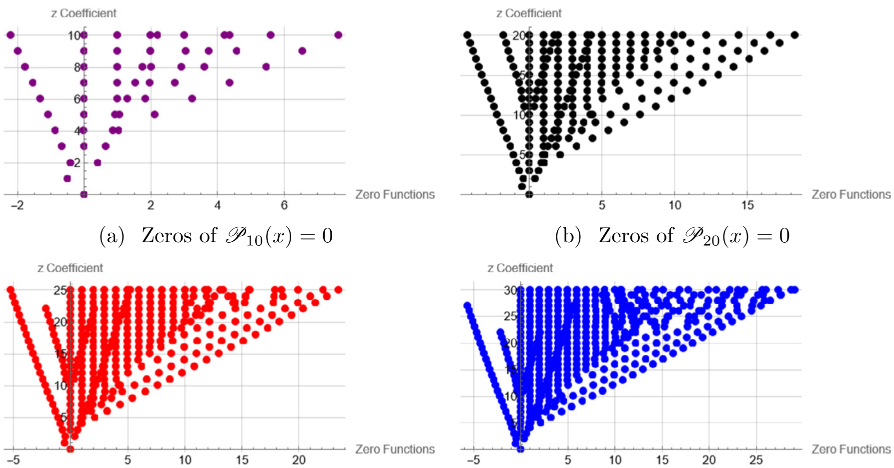
Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of the equation, we obtain:

$$\mathscr{P}_n(x) - \mathscr{P}_n = \sum_{k=0}^{n-1}\frac{n}{(k+1)}\binom{n-1}{k}(x)_{k+1}\mathscr{P}_{n-k-1}$$

$$\mathscr{P}_n(x) = \mathscr{P}_n + \sum_{k=0}^{n-1}\frac{n}{(k+1)}\binom{n-1}{k}(x)_{k+1}\mathscr{P}_{n-k-1}.$$

10.3 Approximate roots of U-Bernoulli–Korobov-type polynomials and their applications

In this section, we investigate certain zero distributions of U-Bernoulli–Korobov-type polynomials. The graph plots using the Mathematica program show the zero distribution patterns.



(a) Zeros of $\mathscr{P}_{10}(x) = 0$ (b) Zeros of $\mathscr{P}_{20}(x) = 0$

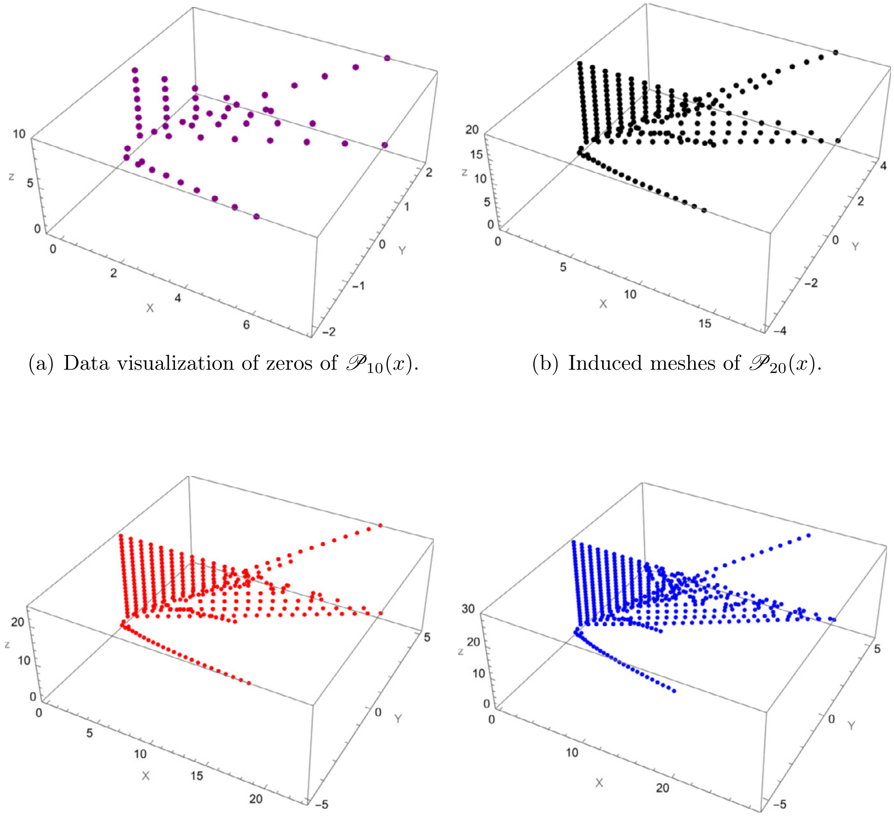
(c) Zeros of $\mathscr{P}_{25}(x) = 0$ (d) Zeros of $\mathscr{P}_{30}(x) = 0$

FIGURE 10.1

Roots of U-Bernoulli–Korobov-type polynomials, plotted considering 10, 20, 25, and 30 points.

In Fig. 10.1 we show four different plots, each representing an amount of zero distribution. In 10.1(a) purple dots for 10 points, in 10.1(b) black dots for 20 points, in 10.1(c) red dots for 25 points, and 1(d) blue dots for 30 points, and we can see that all the roots are located along the x-axes, except in the zero distribution for 10.1(b).

Finally, we will show the induced mesh of U-Bernoulli–Korobov-type polynomials for different values of n (10.5) $\mathscr{P}_n(x) = 0$.



(a) Data visualization of zeros of $\mathscr{P}_{10}(x)$.

(b) Induced meshes of $\mathscr{P}_{20}(x)$.

(c) Data visualization of zeros of $\mathscr{P}_{25}(x)$.

(d) Data visualization of zeros of $\mathscr{P}_{30}(x)$.

FIGURE 10.2

Data visualization of zeros of U-Bernoulli–Korobov-type polynomials, plotted considering 10, 20, 25, and 30 points.

In Fig. 10.2, we present four 3D plots illustrating the distribution of the zeros of the polynomials $\mathscr{P}_n(x)$ for degrees $n = 10$, 20, 25, and 30, respectively. The plots reveal that the zeros tend to cluster toward the upper left region of each mesh. As the degree increases, the number of zeros naturally grows, and the mesh becomes denser and more structured. Notably, a slight perturbation appears around $x \approx 10$, becoming especially noticeable in the blue mesh of Fig. 10.2(d), corresponding to $\mathscr{P}_{30}(x)$.

Next, we calculated an approximate solution satisfying the U-Bernoulli–Korobov-type polynomials $\mathscr{P}_n(x)$ for $n = 2, 3, \ldots, 15$. The results are presented in Table 10.1.

Table 10.1 Approximate solutions for $\mathscr{R}_n(x) = 0$.

Grade n	x
0	—
1	-0.5000
2	$-0.4082i$, $0.4082i$
3	0, $0.75 - 0.6614i$, $0.75 + 0.6614i$
4	-0.01087, 1.033, $1.489 \pm 0.8662i$
5	0, 0.9438, 2.132, $2.212 \pm 1.074i$
6	0.0003813, 0.9977, 1.850, 3.253, $2.949 \pm 1.303i$
7	0, 1.003, 1.982, 2.728, 4.363, $3.712 \pm 1.537i$
8	-0.00001231, 1.000, 2.014, 2.918, 3.620, 5.462 $4.493 \pm 1.763i$
9	0, 0.9999, 2.001, 3.048, 3.751, 4.577 6.553, $5.285 \pm 1.983i$
10	3.821×10^{-7}, 1.000, 1.999, 3.005, 4.218, 4.366 5.605, 7.639, $6.084 \pm 2.2i$

10.4 Conclusions

In this work, the algebraic and differential properties of a new family of polynomials called discrete U-Bernoulli–Korobov polynomials were defined and studied. They are constructed from a generating function that combines characteristics of Bernoulli and Korobov polynomials. Recurrence formulas, explicit expressions, and functional relations describing the behavior of these polynomials were presented, including their expansion in power series, derivatives, and discrete shifts. Additionally, the approximate roots of these polynomials were analyzed using graphical representations generated with computer programs, revealing structured patterns and a characteristic concentration of zeros in certain regions of the plane. These observations allow for a deeper study of the behavior of their zeros and open up new possibilities for their application in mathematical contexts related to discrete analysis and special polynomial theory.

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