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PAPER

The exact solutions for the fractional Riemann wave equation in quantum mechanics and optics

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Abstract

In this paper, the fractional Riemann wave equation with M-truncated derivative (FRWE-MTD) is considered. The Jacobi elliptic function method and the modified extended tanh function method are applied to acquire new elliptic, rational, hyperbolic, and trigonometric functions solutions. Moreover, we expand some earlier studies. The obtained solutions are important in explaining some exciting physical phenomena, since the Riemann wave equation is used in various fields, including quantum mechanics, optics, signal processing, and general relativity. Also, this equation is used to describe the propagation of waves in various dispersive systems, where wave motion is affected by the medium through which it travels. Several 3D and 2D graphs are shown to demonstrate how the M-truncated derivative affects the exact solutions of the FRWE-MTD.

1. Introduction

The fractional nonlinear evolution equation (FNLEE) is a mathematical equation that includes both a nonlinear function and a fractional derivative. Fractional derivatives are a type of ordinary derivative that is appropriate for explaining systems with memory effects or long-range interactions. These equations have recently gained a lot of interest because of their ability to represent and clarify many different physical phenomena [1–5].

Many physical phenomena, including diffusion processes, viscoelasticity, and heat conduction, display nonlinearity and memory effects. Fractional nonlinear evolution equations have been effectively used to describe and comprehend these events, offering more insight into the underlying dynamics. For example, in the study of heat conduction, the traditional Fourier's law is expanded using fractional derivatives to reflect the nonlocal and nonlinear effects in the system.

Furthermore, many engineering systems, including electrical circuits, control systems, and signal processing, have nonlinear and memory-dependent behavior. Traditional linear models are insufficient to effectively describe and analyze complex systems. Fractional nonlinear evolution equations offer an effective foundation for designing and optimizing engineering systems, allowing engineers to create more effective and robust solutions.

Due to the importance of FNLEE, many authors have proposed many methods for obtaining its solutions for instance Lie symmetry method [6], Lie group analysis [7, 8], fractional sub-equation method [9], invariant subspace method [8], Jacobi elliptic function method [10], optimal auxiliary function method [11], generalized Riccati equation method [12], He's semi-inverse method [13], and etc.

Considering fractional differential equations with regard to time can help us comprehend a variety of physical phenomena. Incorporating fractional derivatives into a model allows us to get deeper insights into the behavior of physical systems and design improved control techniques. Therefore, in this paper, we look at the fractional Riemann wave equation with M-truncated derivative (FRWE-MTD) [14]:

$$\begin{aligned} &\mathcal{M}_{\delta,t}^{\alpha,\sigma}\mathcal{W} + \varepsilon_1\mathcal{W}_{xxy} + \varepsilon_2\mathcal{W}\mathcal{R}_x + \varepsilon_3 R W_x = 0,\\ &\mathcal{W}_y = \mathcal{R}_x, \end{aligned} \tag{1}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are real constants and $\mathcal{M}_{\delta,t}^{\alpha,\sigma}$ is M-truncated derivative operator [15]. We consider the M-truncated derivative here because it satisfies all classical derivative formulas including the chain rule, quotient rule, and product rule. Also, it generalizes the conformable fractional derivative with $\delta = 1$.

The Riemann wave equation (RWE) has several applications in a range of scientific disciplines. The RWE is important in quantum mechanics because it characterizes particle behavior at the quantum level. The equation depicts the wave function of a particle, revealing its location, momentum, and energy. Along with quantum mechanics concepts, the Riemann wave equation helps us understand wave-particle duality and quantum tunneling.

In optics, the RWE is essential for understanding the behavior of light waves. It enables scientists and engineers to investigate light characteristics including as refraction, diffraction, and interference, as well as design optical devices such as lenses and cameras.

In acoustics, it aids in the study of sound waves and their propagation across different mediums. Solving the Riemann wave equation allows us to determine the behavior of sound waves in air, water, and other materials, which aids in the design of acoustic systems such as speakers and musical instruments.

Due to the significance of RWE, numerous authors have obtained the solution of the Riemann wave equation by employing various methods, including generalized (G'/G)-expansion method [16], new extended direct algebraic method [17], generalized Kudryashov method [18], extended tanh function technique [19], Wronskian method [20], generalized exponential rational function approach [21], and modified $\exp(-u(\theta))$-function method [22]. While, the solutions of FRWE with conformable fractional derivative have acquired by improved (G'/G)-expansion method [23]. Moreover, the stochastic Riemann wave equation was considered by Mohammed et al. [24] and they found its solutions by using the extended tanh–coth method and the mapping method.

Our novelties of this paper are:

- Obtaining the exact solutions of FRWE-MTD (1). To get these solutions, we apply two different approaches, including the Jacobi elliptic function and modified extended tanh function methods. By using these methods various kinds of solitons such as periodic solitons, dark solitons, bright solitons, kink solitons and another solitons solutions for FRWE-MTD (1) are obtained. Due to the practical value of soliton solutions in the Riemann wave equation in describing certain intriguing scientific phenomena since the RWE equation is employed in a variety of domains, including quantum physics, optics, signal processing, and general relativity.
- Expanding upon some earlier research, such as the results presented in [19].
- Studying the impact of M-truncated derivative on the exact solutions of the FRWE-MTD (1) by plotting various obtained solutions.

The following is the outline of the article: In section 2, the definition and properties of M-truncated derivative are given. In section 3, the wave equation for the FRWE-MTD (1) is derived. In section 4, the exact solution of the FRWE-MTD (1) is acquired. While in section 5, the impact of the M-truncated derivative on the generated solutions of FRWE-MTD is addressed. Finally, we introduce the conclusions of this paper.

2. M-truncated derivative

Several versions of fractional derivatives have been developed throughout the years, each attempt to provide a more accurate and effective description of the fractional order derivatives. The most commonly used versions are the Riemann-Liouville, Grünwald-Letnikov, Caputo and Hadamard, Riesz, Erdelyi [25–28]. Classical derivative formulas, such as the chain rule, quotient rule, and product rule, do not apply to the wide variety of fractional derivative kinds. Recently, Sousa *et al* [15] introduced a new derivative known as M-truncated derivative (MTD) which satisfies all classical derivative formulas. They defined the MTD of order $0 < \alpha \leqslant 1$ for the function u: $[0, \infty) \to \mathbb{R}$ as

$$\mathcal{M}_{\delta,t}^{\alpha,\sigma} u(t) = \lim_{h\to 0} \frac{u(tE_{\delta,\sigma}(ht^{-\alpha})) - u(t)}{h}, \text{ for } t > 0,$$

where

$$E_{\delta,\sigma}(x) = \sum_{k=0}^{\delta} \frac{x^k}{\Gamma(\sigma k + 1)},$$

for $\sigma > 0$ and $x \in \mathbb{C}$.

The MTD has the following properties for all differentiable functions Z and ψ and for real constants a, b, υ:

(1) $\mathcal{M}_{\delta,t}^{\alpha,\sigma}(aZ + b\psi) = a\mathcal{M}_{\delta,t}^{\alpha,\sigma}(Z) + b\mathcal{M}_{\delta,t}^{\alpha,\sigma}(\psi)$;

(2) $\mathcal{M}_{\delta,t}^{\alpha,\sigma}(t^{\nu}) = \frac{\nu}{\Gamma(\sigma+1)} t^{\nu-\alpha}$;

(3) $\mathcal{M}_{\delta,t}^{\alpha,\sigma}(Z\psi) = Z\mathcal{M}_{\delta,t}^{\alpha,\sigma}\psi + \psi\mathcal{M}_{\delta,t}^{\alpha,\sigma}Z$;

(4) $\mathcal{M}_{\delta,t}^{\alpha,\sigma}(Z)(t) = \frac{t^{1-\alpha}}{\Gamma(\sigma+1)}\frac{dZ}{dt}$;

(5) $\mathcal{M}_{\delta,t}^{\alpha,\sigma}(Z\circ\psi)(t) = Z'(\psi(t))\mathcal{M}_{\delta,t}^{\alpha,\sigma}\psi(t)$.

3. Wave equation for the FRWE-MTD

To deduce the wave equation of FRWE-MTD (1), we apply

$$\mathcal{W}(x, y, t) = u(\theta),\ \mathcal{R}(x, y, t) = v(\theta) \text{and } \theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)\omega}{\alpha} t^{\alpha}, \tag{2}$$

where u and v are real functions, θ_1 and θ_2 are the wave frequency, and ω is the wave speed. It is noted that

$$\mathcal{W}_x = \theta_1 u',\ \mathcal{W}_y = \theta_2 u',\ \mathcal{W}_{xxy} = \theta_1^2\theta_2 u''',\ \mathcal{R}_x = \theta_1 v', \tag{3}$$

and

$$\mathcal{M}_{\delta,t}^{\alpha,\sigma}\mathcal{W} = \omega u'. \tag{4}$$

Inserting equation (2) into equation (1) and utilizing (3–4), we obtain

$$\begin{aligned} &\omega u' + \varepsilon_1\theta_1^2\theta_2 u''' + (\varepsilon_2\theta_1 uv' + \varepsilon_3\theta_1 vu') = 0,\\ &\theta_2 u' = \theta_1 v'. \end{aligned} \tag{5}$$

Integrating the second equation of (5) and neglecting the integral constant, we get

$$v = \frac{\theta_2}{\theta_1}u. \tag{6}$$

Plugging equation (6) into the first equation of (5), we have

$$\varepsilon_1\theta_1^2\theta_2 u''' + \omega u' + (\varepsilon_2\theta_2 + \varepsilon_3\theta_2)uu' = 0. \tag{7}$$

Integrating equation (7) once, we attain

$$u'' + \hbar_1 u + \hbar_2 u^2 = 0, \tag{8}$$

where

$$\hbar_1 = \frac{\omega}{\varepsilon_1\theta_1^2\theta_2} \text{ and } \hbar_2 = \frac{\varepsilon_2 + \varepsilon_3}{2\varepsilon_1\theta_1^2}.$$

4. Exact solutions of FRWE-MTD

To get exact solutions for equation (1), we use two various methods: Jacobi elliptic function method (JEF-method) and the modified extended tanh function method (METF-method).

4.1. JEF-method

Here, we apply the JEF-method described in [29]. Let the solutions of equation (8) have the form:

$$u(\theta) = \sum_{j=0}^{M}\gamma_k\Omega^k(\theta), \tag{9}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_M$ are undefined constants such that $\gamma_M \neq 0$ and $\Omega(\theta) = sn(\theta, \delta)$ is Jacobi elliptic sine function for $0 < \delta < 1$. To find the value of M, we balance u^2 with u'' in equation (8) to have

$$2M = M + 2,$$

then

$$M = 2. \tag{10}$$

Rewriting equation (9) and using equation (10), we get

$$u(\theta) = \gamma_0 + \gamma_1\Omega(\theta) + \gamma_2\Omega^2(\theta). \tag{11}$$

Plugging equation (11) into equation (8), we have

$$(6\delta^2\gamma_2 + \hbar_2\gamma_2^2)\Omega^4 + (2\delta^2\gamma_1 + 2\hbar_2\gamma_1\gamma_2)\Omega^3 + (2\gamma_0\gamma_2\hbar_2 - 4\gamma_2(\delta^2 + 1) + \hbar_1\gamma_2 + \hbar_2\gamma_1^2)\Omega^2$$
$$-[(\delta^2 + 1)\gamma_1 - \hbar_1\gamma_1 - 2\hbar_2\gamma_0\gamma_1]\Omega + (2\gamma_2 + \hbar_1\gamma_0 + \hbar_2\gamma_0^2) = 0.$$

For $n = 4, 3, 2, 1, 0$, equating each coefficient of Ω^n to zero, we attain

$$6\delta^2\gamma_2 + \hbar_2\gamma_2^2 = 0,$$
$$2\delta^2\gamma_1 + 2\hbar_2\gamma_1\gamma_2 = 0,$$
$$2\gamma_0\gamma_2\hbar_2 - 4\gamma_2(\delta^2 + 1) + \hbar_1\gamma_2 + \hbar_2\gamma_1^2 = 0,$$
$$(\delta^2 + 1)\gamma_1 - \hbar_1\gamma_1 - 2\hbar_2\gamma_0\gamma_1 = 0,$$

and

$$2\gamma_2 + \hbar_1\gamma_0 + \hbar_2\gamma_0^2 = 0.$$

The solutions of these equations yields the following two sets:

First set:

$$\begin{cases} \gamma_0 = \dfrac{4(\delta^2+1) - \sqrt{16\delta^4 - 16\delta^2 + 1}}{2\hbar_2}, \\ \gamma_1 = 0, \\ \gamma_2 = \dfrac{-6\delta^2}{\hbar_2}, \\ \omega = 4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4 - \delta^2 + 1}. \end{cases}$$

Second set:

$$\begin{cases} \gamma_0 = \dfrac{2(\delta^2+1) + 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2},, \\ \gamma_1 = 0, \\ \gamma_2 = \dfrac{-6\delta^2}{\varepsilon_2}, \\ \omega = -4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4 - \delta^2 + 1}. \end{cases}$$

For the first set, the solutions of FRWE-MTD (1), utilizing (11), are

$$\mathcal{W}(x, y, t) = \frac{2(\delta^2+1) - 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} - \frac{6\delta^2}{\hbar_2} sn^2(\theta, \delta), \tag{12}$$

$$\mathcal{R}(x, y, t) = \frac{\theta_2}{\theta_1}\left[\frac{2(\delta^2+1) - 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} - \frac{6\delta^2}{\hbar_2} sn^2(\theta, \delta)\right], \tag{13}$$

where $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4 - \delta^2 + 1}]}{\alpha} t^\alpha$. When $\delta \rightarrow 1$, equations (12) and (13) become

$$\mathcal{W}(x, y, t) = \frac{2}{\hbar_2} - \frac{6}{\hbar_2}\tanh^2(\theta), \tag{14}$$

$$\mathcal{R}(x, y, t) = \frac{\theta_2}{\theta_1}\left[\frac{2}{\hbar_2} - \frac{6}{\hbar_2}\tanh^2(\theta)\right], \tag{15}$$

with $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[\varepsilon_1\theta_1^2\theta_2]}{\alpha} t^\alpha$.

For the second set, the solutions of FRWE-MTD (1), utilizing (11), are

$$\mathcal{W}(x, y, t) = \frac{2(\delta^2+1) + 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} - \frac{6\delta^2}{\hbar_2} sn^2(\theta, \delta), \tag{16}$$

$$\mathcal{R}(x, y, t) = \frac{\theta_2}{\theta_1}\left[\frac{2(\delta^2+1) + 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} - \frac{6\delta^2}{\hbar_2} sn^2(\theta, \delta)\right], \tag{17}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4 - \delta^2 + 1}]}{\alpha} t^\alpha$.

When $\delta \rightarrow 1$, equations (16) and (17) become

$$\mathcal{W}(x, y, t) = \frac{6}{\hbar_2} - \frac{6}{\hbar_2}\tanh^2(\theta) = \frac{6}{\hbar_2}\operatorname{sech}^2(\theta), \tag{18}$$

$$\mathcal{R}(x, y, t) = \frac{6\theta_2}{\theta_1\hbar_2}\operatorname{sech}^2(\theta), \tag{19}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[4\varepsilon_1\theta_1^2\theta_2]}{\alpha} t^\alpha$.

In similar steps, we can change sn in (11) by cn, where $cn(\theta, \delta)$ is Jacobi elliptic cosine function, to have the FRWE-MTD (1) as follows:

$$\mathcal{W}(x, y, t) = \frac{(2 - 4\delta^2) - 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} + \frac{6\delta^2}{\hbar_2} cn^2(\theta, \delta), \tag{20}$$

$$\mathcal{R}(x, y, t) = \frac{\theta_2}{\theta_1}\left[\frac{(2 - 4\delta^2) - 2\sqrt{\delta^4 - \delta^2 + 1}}{\hbar_2} + \frac{6\delta^2}{\hbar_2} cn^2(\theta, \delta)\right], \tag{21}$$

where $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4-\delta^2+1}]}{\alpha}t^{\alpha}$, or

$$\mathcal{W}(x, y, t) = \frac{(2-4\delta^2)+2\sqrt{\delta^4-\delta^2+1}}{\hbar_2} + \frac{6\delta^2}{\hbar_2}cn^2(\theta, \delta), \tag{22}$$

$$\mathcal{R}(x, y, t) = \frac{\theta_2}{\theta_1}\left[\frac{(2-4\delta^2)+2\sqrt{\delta^4-\delta^2+1}}{\hbar_2} + \frac{6\delta^2}{\hbar_2}cn^2(\theta, \delta)\right], \tag{23}$$

with $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[4\varepsilon_1\theta_1^2\theta_2\sqrt{\delta^4-\delta^2+1}]}{\alpha}t^{\alpha}$.

4.2. METF-method

We utilize here the METF-method that stated in [30]. Assuming the solution u of equation (8) has the type

$$u(\theta) = \sum_{j=0}^{M=2} a_j Z^j + \sum_{j=1}^{M=2} b_j Z^{-j} = a_0 + a_1 Z + a_2 Z^2 + b_1 Z^{-1} + b_2 Z^{-2}, \tag{24}$$

where Z solves

$$Z' = Z^2 + k. \tag{25}$$

equation (25) has the solutions:

$$Z(\theta) = \sqrt{k}\tan(\sqrt{k}\theta) \text{or } Z(\theta) = -\sqrt{k}\cot(\sqrt{k}\theta) \text{ if } k > 0, \tag{26}$$

or

$$Z(\theta) = -\sqrt{-k}\tanh(\sqrt{-k}\theta) \text{or } Z(\theta) = -\sqrt{-k}\coth(\sqrt{-k}\theta) \text{ if } k < 0, \tag{27}$$

or

$$Z(\theta) = \frac{-1}{\theta} \text{ if } k = 0. \tag{28}$$

Substituting equation (24) into equation (8), we get

$$\begin{aligned}
&(6a_2 + \hbar_2 a_2^2)Z^4 + (2a_1 + 2\hbar_2 a_1 a_2)Z^3 + (8ka_2 + 2a_0 a_2\hbar_2 + a_1^2\hbar_2 + \hbar_1 a_2)Z^2 \\
&(2ka_1 + \hbar_1 a_1 + 2\hbar_2 a_0 a_1 + 2a_2 b_1)Z + (2k^2 a_2 + 2b_2 + \hbar_1 a_0 + \hbar_2 a_0^2 + 2\hbar_2 a_1 b_1 \\
&+2\hbar_2 a_2 b_2) + (2kb_1 + 2\hbar_2 a_0 b_1 + 2\hbar_2 a_1 b_2 + \hbar_1 b_1)Z^{-1} + (8kb_2 + 2a_0 b_2\hbar_2 \\
&+b_1^2\hbar_2 + \hbar_1 b_2)Z^{-2} + (2b_1 k^2 + 2\hbar_2 b_1 b_2)Z^{-3} + (6k^2 b_2 + \hbar_2 b_2^2)Z^4 = 0.
\end{aligned}$$

The coefficients of each power of Z are set to zero as follows:

$$\begin{aligned}
6a_2 + \hbar_2 a_2^2 &= 0, \\
2a_1 + 2\hbar_2 a_1 a_2 &= 0, \\
8ka_2 + 2a_0 a_2\hbar_2 + a_1^2\hbar_2 + \hbar_1 a_2 &= 0, \\
2ka_1 + \hbar_1 a_1 + 2\hbar_2 a_0 a_1 + 2a_2 b_1 &= 0, \\
2k^2 a_2 + 2b_2 + \hbar_1 a_0 + \hbar_2 a_0^2 + 2\hbar_2 a_1 b_1 + 2\hbar_2 a_2 b_2 &= 0, \\
2kb_1 + 2\hbar_2 a_0 b_1 + 2\hbar_2 a_1 b_2 + \hbar_1 b_1 &= 0, \\
8kb_2 + 2a_0 b_2\hbar_2 + b_1^2\hbar_2 + \hbar_1 b_2 &= 0, \\
2b_1 k^2 + 2\hbar_2 b_1 b_2 &= 0.
\end{aligned}$$

and

$$6k^2 b_2 + \hbar_2 b_2^2 = 0.$$

We get the four separate sets by solving these equations as follows:

First set:

$$a_0 = \frac{-6k}{\hbar_2}, a_1 = 0, a_2 = \frac{-6}{\hbar_2}, \quad b_1 = 0, b_2 = 0, \omega = 4k\varepsilon_1\theta_1^2\theta_2. \tag{29}$$

Second set:

$$a_0 = \frac{-2k}{\hbar_2}, a_1 = 0, a_2 = \frac{-6}{\hbar_2}, \quad b_1 = 0, b_2 = 0, \omega = -4k\varepsilon_1\theta_1^2\theta_2. \tag{30}$$

Third set:

$$a_0 = \frac{-12k}{\hbar_2},\ a_1 = 0,\ a_2 = \frac{-6}{\hbar_2},\ \ b_1 = 0,\ b_2 = \frac{-6k^2}{\hbar_2},\ \omega = 16k\varepsilon_1\theta_1^2\theta_2. \tag{31}$$

Fourth set:

$$a_0 = \frac{8k}{\hbar_2},\ a_1 = 0,\ a_2 = \frac{-6}{\hbar_2},\ \ b_1 = 0,\ b_2 = \frac{-6k^2}{\hbar_2},\ \omega = -14k\varepsilon_1\theta_1^2\theta_2. \tag{32}$$

First set: By using (29), the solution of equation (8) takes the form

$$u(\theta) = \frac{-6k}{\hbar_2} - \frac{6}{\hbar_2}Z^2(\theta).$$

For $Z(\theta)$, there are many distinct cases:

Case1: When $k > 0$, we have by using (26)

$$u(\theta) = \frac{-6k}{\hbar_2} - \frac{6k}{\hbar_2}\tan^2(\sqrt{k}\,\theta) = -\frac{6k}{\hbar_2}\sec^2(\sqrt{k}\,\theta),$$

and

$$u(\theta) = \frac{-6k}{\hbar_2} - \frac{6k}{\hbar_2}\cot^2(\sqrt{k}\,\theta) = \frac{-6k}{\hbar_2}\csc^2(\sqrt{k}\,\theta).$$

Hence, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x, y, t) = -\frac{6k}{\hbar_2}\sec^2(\sqrt{k}\,\theta), \tag{33}$$

$$\mathcal{R}(x, y, t) = -\frac{6k\theta_2}{\hbar_2\theta_1}\sec^2(\sqrt{k}\,\theta), \tag{34}$$

and

$$\mathcal{W}(x, y, t) = \frac{-6k}{\hbar_2}\csc^2(\sqrt{k}\,\theta), \tag{35}$$

$$\mathcal{R}(x, y, t) = -\frac{6k\theta_2}{\hbar_2\theta_1}\csc^2(\sqrt{k}\,\theta), \tag{36}$$

where $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[4k\varepsilon_1\theta_1^2\theta_2]}{\alpha}t^\alpha$.

Case2: When $k < 0$, we have by using (27)

$$u(\theta) = \frac{-6k}{\hbar_2} + \frac{6k}{\hbar_2}\tanh^2(\sqrt{-k}\,\theta) = \frac{-6k}{\hbar_2}\operatorname{sech}^2(\sqrt{-k}\,\theta),$$

and

$$u(\theta) = \frac{-6k}{\hbar_2} + \frac{6k}{\hbar_2}\coth^2(\sqrt{-k}\,\theta) = \frac{6k}{\hbar_2}\operatorname{csch}^2(\sqrt{-k}\,\theta).$$

Hence, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x, y, t) = \frac{-6k}{\hbar_2}\operatorname{sech}^2(\sqrt{-k}\,\theta), \tag{37}$$

$$\mathcal{R}(x, y, t) = -\frac{6k\theta_2}{\hbar_2\theta_1}\operatorname{sech}^2(\sqrt{-k}\,\theta), \tag{38}$$

and

$$\mathcal{W}(x, y, t) = \frac{6k}{\hbar_2}\operatorname{csch}^2(\sqrt{-k}\,\theta), \tag{39}$$

$$\mathcal{R}(x, y, t) = \frac{6k\theta_2}{\hbar_2\theta_1}\operatorname{csch}^2(\sqrt{-k}\,\theta). \tag{40}$$

Case3: When $k = 0$, we get by using (28)

$$u(\theta) = -\frac{6}{\hbar_2}\frac{1}{\theta^2}.$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\, y,\, t) = \frac{-6}{\hbar_2 \theta^2}, \tag{41}$$

$$\mathcal{R}(x,\, y,\, t) = \frac{-6\theta_2}{\hbar_2 \theta_1 \theta^2}, \tag{42}$$

where $\theta = \theta_1 x + \theta_2 y$.

Second set: By using (30), the solution of equation (8) takes the form

$$u(\theta) = \frac{-2k}{\hbar_2} - \frac{6}{\hbar_2} Z^2(\theta).$$

For $Z(\theta)$, there are many distinct cases:

Case1: When $k > 0$, we have by using (26)

$$u(\theta) = \frac{-2k}{\hbar_2} - \frac{6k}{\hbar_2} \tan^2(\sqrt{k}\,\theta),$$

and

$$u(\theta) = \frac{-2k}{\hbar_2} - \frac{6k}{\hbar_2} \cot^2(\sqrt{k}\,\theta).$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\, y,\, t) = \frac{-2k}{\hbar_2} - \frac{6k}{\hbar_2} \tan^2(\sqrt{k}\,\theta), \tag{43}$$

$$\mathcal{R}(x,\, y,\, t) = \frac{-2k\theta_2}{\hbar_2 \theta_1} - \frac{6k\theta_2}{\hbar_2 \theta_1} \tan^2(\sqrt{k}\,\theta), \tag{44}$$

and

$$\mathcal{W}(x,\, y,\, t) = \frac{-2k}{\hbar_2} - \frac{6k}{\hbar_2} \cot^2(\sqrt{k}\,\theta), \tag{45}$$

$$\mathcal{R}(x,\, y,\, t) = \frac{-2k\theta_2}{\hbar_2 \theta_1} - \frac{6k\theta_2}{\hbar_2 \theta_1} \cot^2(\sqrt{k}\,\theta), \tag{46}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[4k\varepsilon_1 \theta_1^2 \theta_2]}{\alpha} t^{\alpha}$.

Case2: When $k < 0$, one can get by using (27)

$$u(\theta) = \frac{-2k}{\hbar_2} + \frac{6k}{\hbar_2} \tanh^2(\sqrt{-k}\,\theta),$$

and

$$u(\theta) = \frac{-2k}{\hbar_2} + \frac{6k}{\hbar_2} \coth^2(\sqrt{-k}\,\theta).$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\, y,\, t) = \frac{-2k}{\hbar_2} + \frac{6k}{\hbar_2} \tanh^2(\sqrt{-k}\,\theta), \tag{47}$$

$$\mathcal{R}(x,\, y,\, t) = \frac{-2k\theta_2}{\hbar_2 \theta_1} + \frac{6k\theta_2}{\hbar_2 \theta_1} \tanh^2(\sqrt{k}\,\theta), \tag{48}$$

and

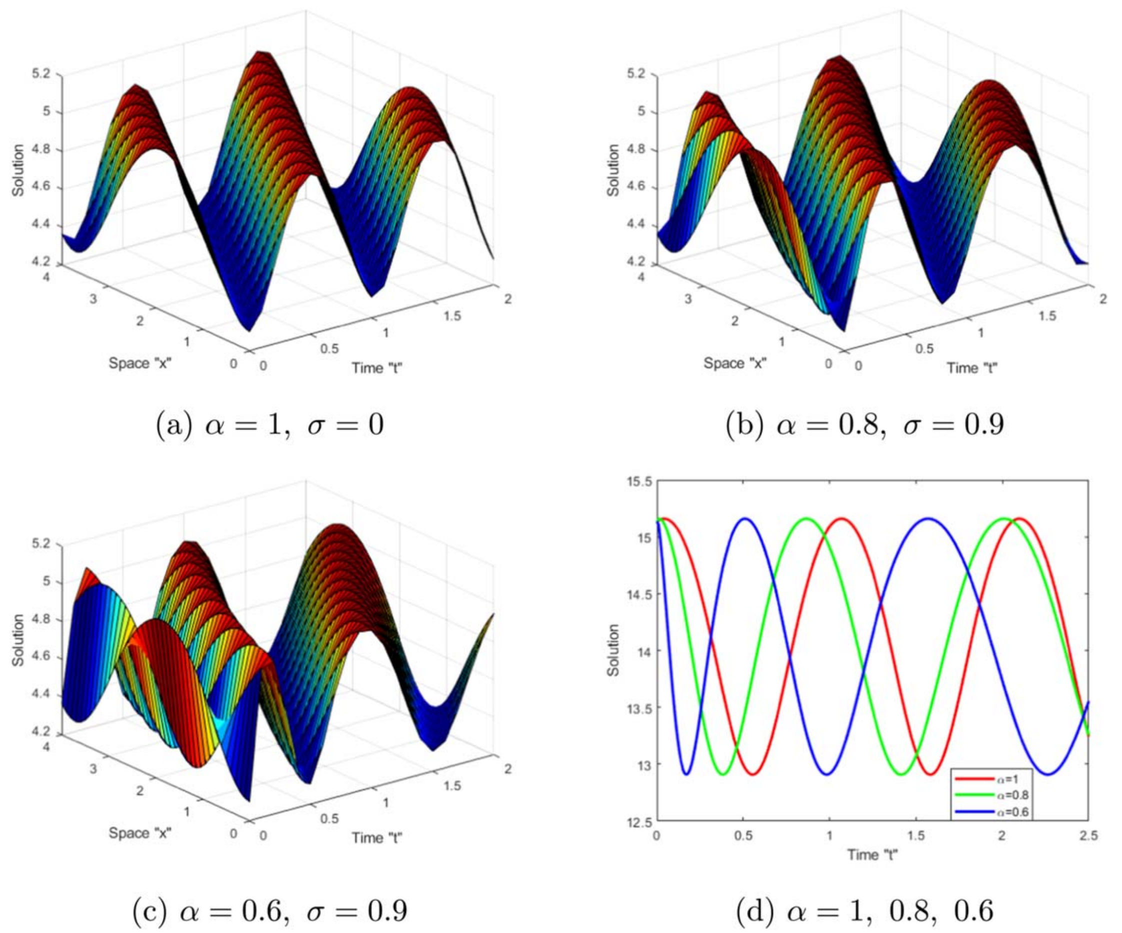
$$\mathcal{W}(x,\, y,\, t) = \frac{-2k}{\hbar_2} + \frac{6k}{\hbar_2} \coth^2(\sqrt{-k}\,\theta), \tag{49}$$

$$\mathcal{R}(x,\, y,\, t) = \frac{-2k\theta_2}{\hbar_2 \theta_1} + \frac{6k\theta_2}{\hbar_2 \theta_1} \coth^2(\sqrt{k}\,\theta), \tag{50}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[4k\varepsilon_1 \theta_1^2 \theta_2]}{\alpha} t^{\alpha}$.

Case 3: When $k = 0$, we have by using (28)

$$u(\theta) = \frac{-6}{\hbar_2} \frac{1}{\theta^2}.$$



(a) $\alpha = 1,\ \sigma = 0$ (b) $\alpha = 0.8,\ \sigma = 0.9$ (c) $\alpha = 0.6,\ \sigma = 0.9$ (d) $\alpha = 1,\ 0.8,\ 0.6$

Figure 1. (a-c) present 3D-profile of the periodic solution $\mathcal{W}(x,\ y,\ t)$ given in equation (16) for $\alpha = 1, 0.8, 0.6$ (d) shows 2D-profile for these values of α

Consequently, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\ y,\ t) = \frac{-6}{\hbar_2\theta^2}, \tag{51}$$

$$\mathcal{R}(x,\ y,\ t) = \frac{-6\theta_2}{\hbar_2\theta_1\theta^2}, \tag{52}$$

where $\theta = \theta_1 x + \theta_2 y$.

Third set: By using (31), the solution of equation (8) takes the form

$$u(\theta) = \frac{-12k}{\hbar_2} - \frac{6}{\hbar_2}Z^2(\theta) - \frac{6k^2}{\hbar_2}Z^{-2}(\theta).$$

For $Z(\theta)$, there are many distinct cases:

Case 1: When $k > 0$, we have by using (26)

$$u(\theta) = \frac{-12k}{\hbar_2} - \frac{6k}{\hbar_2}\tan^2(\sqrt{k}\,\theta) - \frac{6k}{\hbar_2}\cot^2(\sqrt{k}\,\theta)$$
$$= -\frac{6k}{\hbar_2}[\sec^2(\sqrt{k}\,\theta) + \csc^2(\sqrt{k}\,\theta)].$$

Consequently, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\ y,\ t) = -\frac{6k}{\hbar_2}[\sec^2(\sqrt{k}\,\theta) + \csc^2(\sqrt{k}\,\theta)], \tag{53}$$

$$\mathcal{R}(x,\ y,\ t) = \frac{-6k\theta_2}{\hbar_2\theta_1}\sec^2(\sqrt{k}\,\theta) - \frac{6k\theta_2}{\hbar_2\theta_1}\csc^2(\sqrt{k}\,\theta), \tag{54}$$

where $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[16k\varepsilon_1\theta_1^2\theta_2]}{\alpha}t^{\alpha}$.

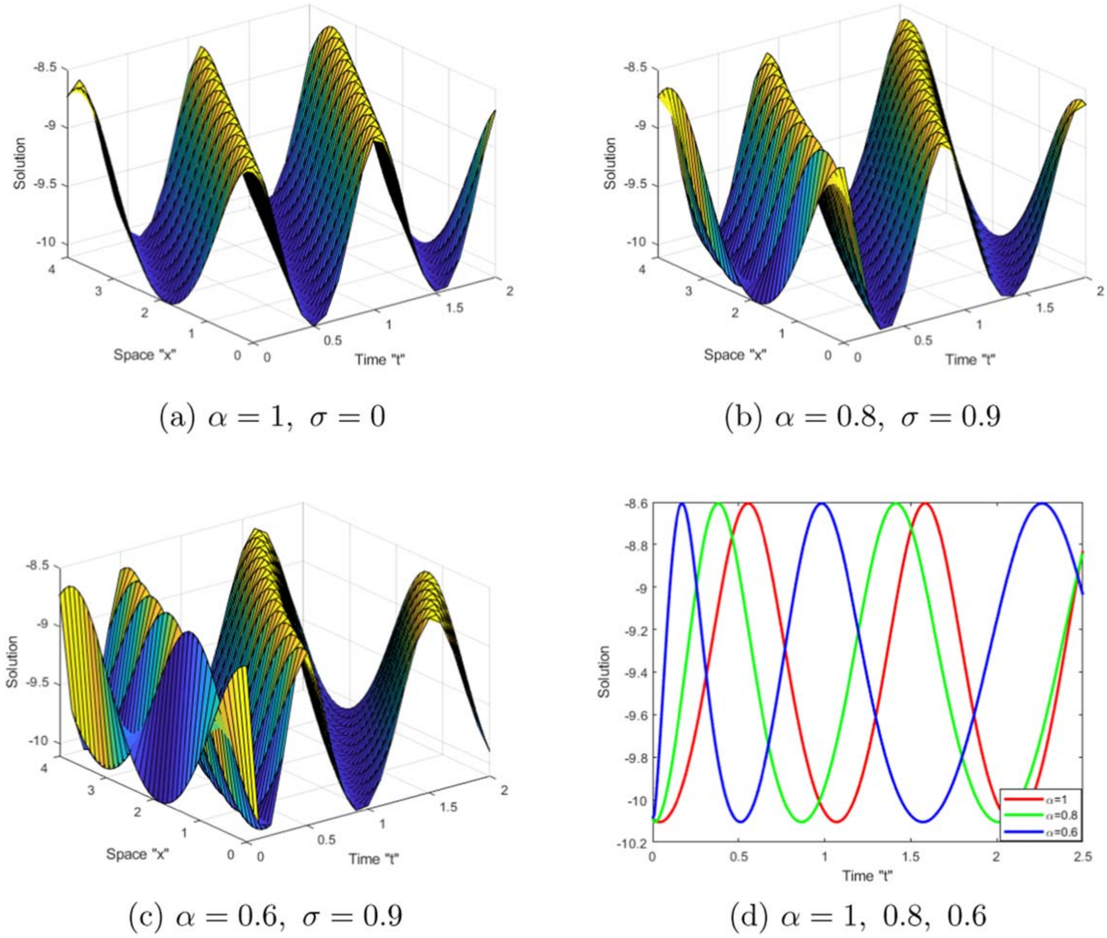


Figure 2. (a-c) present 3D-profile of the periodic solution $\mathcal{R}(x, y, t)$ given in equation (17) for $\theta_2 = -2$ and for $\alpha = 1, 0.8, 0.6$ (d) shows 2D-shape for these values of α

Case 2: When $k < 0$, we get by using (27)

$$\begin{aligned} u(\theta) &= \frac{-12k}{\hbar_2} + \frac{6k}{\hbar_2}\tanh^2(\sqrt{-k}\,\theta) + \frac{6k}{\hbar_2}\coth^2(\sqrt{-k}\,\theta) \\ &= \frac{-6k}{\hbar_2}[\mathrm{sech}^2(\sqrt{-k}\,\theta) - \mathrm{csch}^2(\sqrt{-k}\,\theta)]. \end{aligned}$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x, y, t) = \frac{-6k}{\hbar_2}[\mathrm{sech}^2(\sqrt{-k}\,\theta) - \mathrm{csch}^2(\sqrt{-k}\,\theta)], \tag{55}$$

$$\mathcal{R}(x, y, t) = \frac{-6k\theta_2}{\hbar_2\theta_1}\mathrm{sech}^2(\sqrt{-k}\,\theta) + \frac{6k\theta_2}{\hbar_2\theta_1}\mathrm{csch}^2(\sqrt{-k}\,\theta), \tag{56}$$

where $\theta = \theta_1 x + \theta_2 y + \frac{\Gamma(\sigma+1)[16k\varepsilon_1\theta_1^2\theta_2]}{\alpha}t^\alpha$.

Case 3: When $k = 0$, we have by using (28)

$$u(\theta) = \frac{6}{\hbar_2}\frac{1}{\theta^2} + \frac{6}{\hbar_2}\theta^2.$$

Consequently, the exact solution of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x, y, t) = \frac{6}{\hbar_2}\left[\frac{1}{\theta^2} + \theta^2\right], \tag{57}$$

$$\mathcal{R}(x, y, t) = \frac{6\theta_2}{\hbar_2\theta_1}\left[\frac{1}{\theta^2} + \theta^2\right]. \tag{58}$$

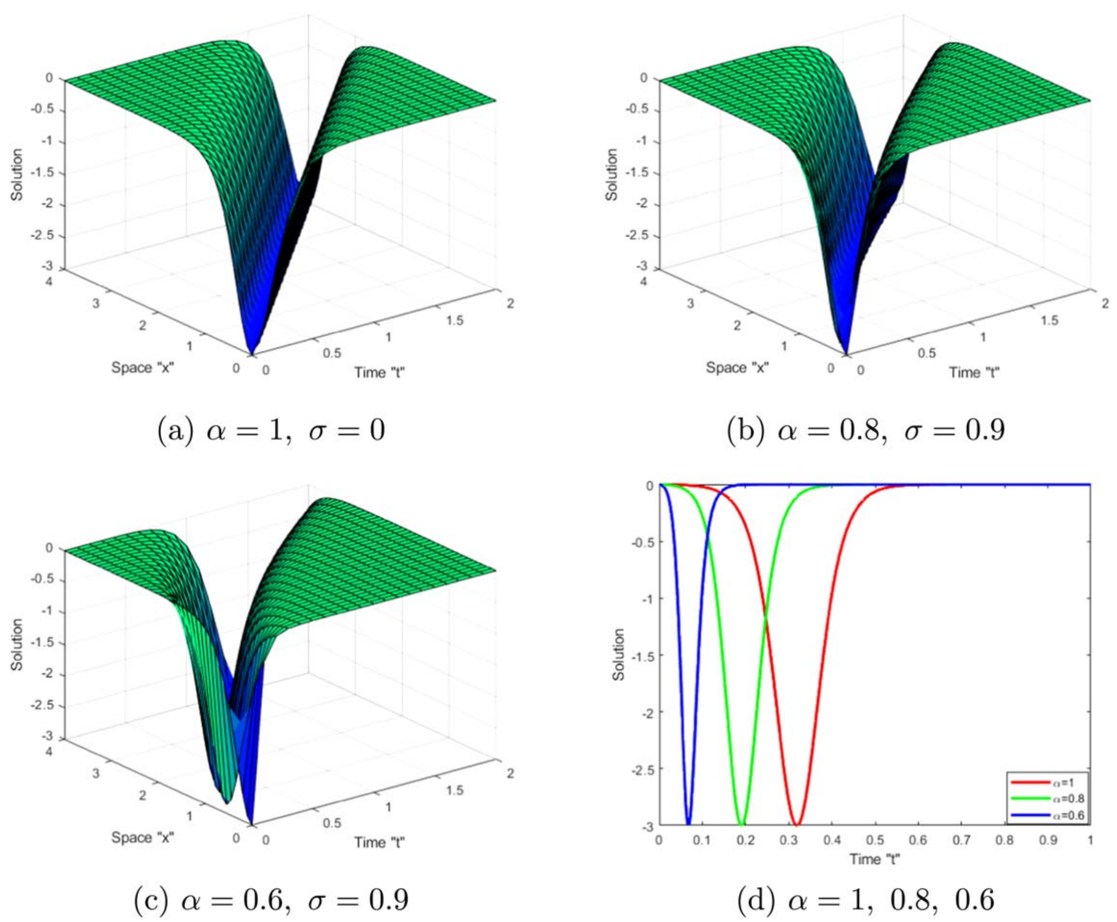


Figure 3. (a-c) present 3D-shape of the dark solution $\mathcal{W}(x,\ y,\ t)$ given in equation (18) for $\delta = 0.5, \theta_2 = -0.8$ and $\alpha = 1, 0.8, 0.6$ (d) shows 2D-profile for these values of α

Fourth set: By using (32), the solution of equation (8) takes the form

$$u(\theta) = \frac{8k}{\hbar_2} - \frac{6}{\hbar_2}Z^2(\theta) - \frac{6k^2}{\hbar_2}Z^{-2}(\theta).$$

For $Z(\theta)$, there are many distinct cases:

Case 1: When $k > 0$, we get by using (26)

$$u(\theta) = \frac{8k}{\hbar_2} - \frac{6k}{\hbar_2}\tan^2(\sqrt{k}\,\theta) - \frac{6k}{\hbar_2}\cot^2(\sqrt{k}\,\theta).$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\ y,\ t) = \frac{8k}{\hbar_2} - \frac{6k}{\hbar_2}\tan^2(\sqrt{k}\,\theta) - \frac{6k}{\hbar_2}\cot^2(\sqrt{k}\,\theta), \tag{59}$$

$$\mathcal{R}(x,\ y,\ t) = \frac{8k\theta_2}{\hbar_2\theta_1} - \frac{6k\theta_2}{\hbar_2\theta_1}\tan^2(\sqrt{k}\,\theta) - \frac{6k\theta_2}{\hbar_2\theta_1}\cot^2(\sqrt{k}\,\theta), \tag{60}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[14k\varepsilon_1\theta_1^2\theta_2]}{\alpha}t^{\alpha}$.

Case 2: When $k < 0$, we have by using (27)

$$u(\theta) = \frac{8k}{\hbar_2} + \frac{6k}{\hbar_2}\tanh^2(\sqrt{-k}\,\theta) + \frac{6k}{\hbar_2}\coth^2(\sqrt{-k}\,\theta).$$

Therefore, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x,\ y,\ t) = \frac{8k}{\hbar_2} + \frac{6k}{\hbar_2}\tanh^2(\sqrt{-k}\,\theta) + \frac{6k}{\hbar_2}\coth^2(\sqrt{-k}\,\theta), \tag{61}$$

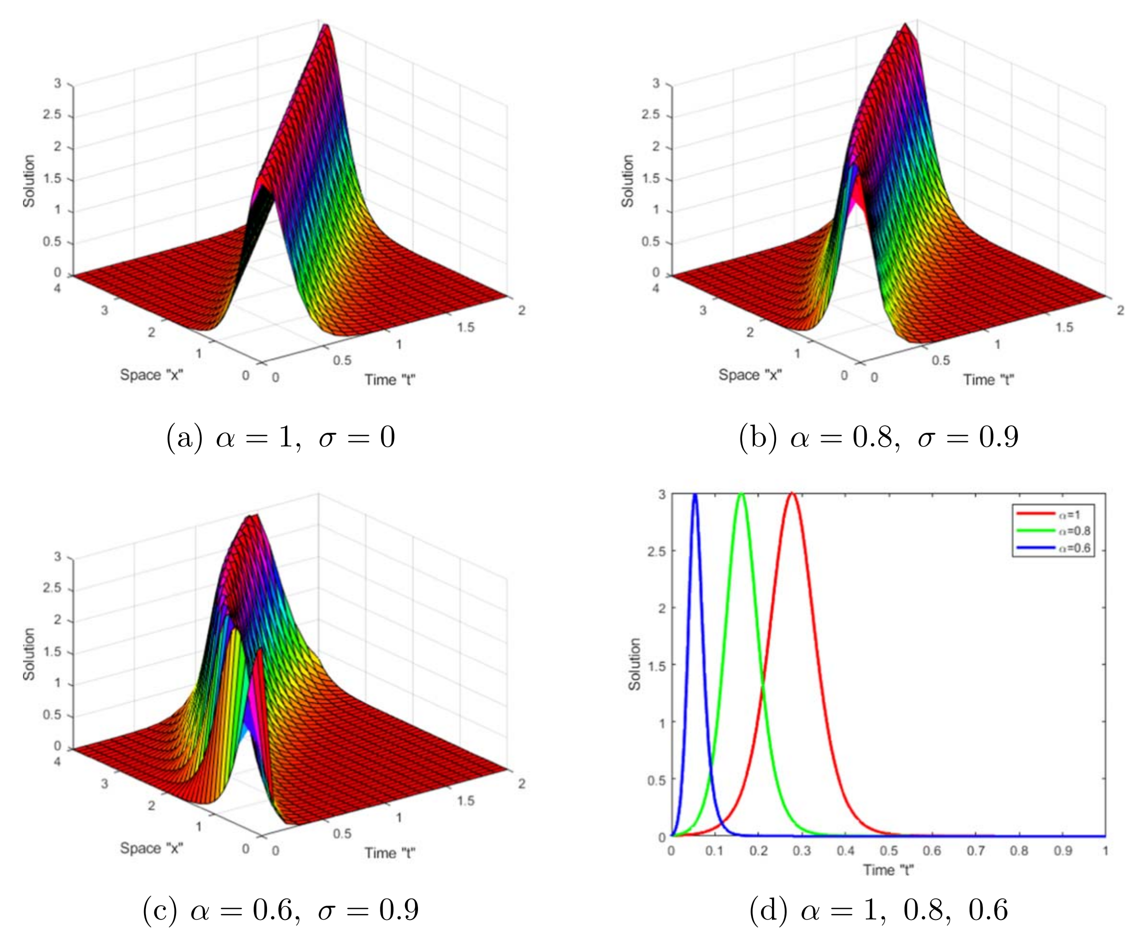


Figure 4. (a-c) present 3D-shape of the bright solution $\mathcal{R}(x, y, t)$ given in equation (19) for $\delta = 0.5, \theta_2 = -0.8$ and for $\alpha = 1, 0.8, 0.6$ (d) shows 2D-profile for these values of α

$$\mathcal{R}(x, y, t) = \frac{8k\theta_2}{\hbar_2\theta_1} + \frac{6k\theta_2}{\hbar_2\theta_1}\tanh^2(\sqrt{-k}\,\theta) + \frac{6k\theta_2}{\hbar_2\theta_1}\coth^2(\sqrt{-k}\,\theta), \tag{62}$$

where $\theta = \theta_1 x + \theta_2 y - \frac{\Gamma(\sigma+1)[14k\varepsilon_1\theta_1^2\theta_2]}{\alpha}t^\alpha$.

Case 3: When $k = 0$, we get by using (28)

$$u(\theta) = \frac{6}{\hbar_2}\left[\frac{1}{\theta^2} + \theta^2\right].$$

Consequently, the exact solutions of FRWE-MTD (1), using (6), are

$$\mathcal{W}(x, y, t) = \frac{6}{\hbar_2}\left[\frac{1}{\theta^2} + \theta^2\right], \tag{63}$$

$$\mathcal{R}(x, y, t) = \frac{6\theta_2}{\hbar_2\theta_1}\left[\frac{1}{\theta^2} + \theta^2\right], \tag{64}$$

where $\theta = \theta_1 x + \theta_2 y$.

Remark 1. If we put $\sigma = 0$ and $\alpha = 1$ in equations (37), (38), (47), (48), (49), (50), (61), and (62), then we attain the same solutions (30), (31), (32), (33), (34), (35), (28), and (29), respectively, that stated in [19].

5. Graphical representations and discussion

In this paper, we acquired the exact solutions of FRWE-MTD (1) by utilizing two various techniques such as JEF-method and METF-method. The JEF-method has provided elliptic soliton solutions. While METF-method has provided rational, hyperbolic, and trigonometric soliton solutions. To illustrate the behavior of these solutions and the impact of the M-truncated derivative on the obtained solutions of FRWE-MTD (1), different graphical representations are provided. For $\varepsilon_1 = -0.5, \varepsilon_2 = \varepsilon_3 = \theta_1 = 1, \theta_2 = -2$ and for varying values of α, we simulate the graphical representations for some attained solutions, including equations (16), (17), (18) and (19) as follows:

From previous figures 1–4, we deduce that when the fractional derivative order α of FRWE-MTD (1) increases, the surface moves into the right side.

6. Conclusions

In this paper, we considered the Riemann wave equation with M-truncated derivative (FRWE-MTD) (1). To get novel elliptic, rational, hyperbolic, and trigonometric function solutions, the Jacobi elliptic function approach and the modified extended tanh function method are used. The performance of modified extended tanh function method is reliable and effective, and it can be applied to many other nonlinear evolution equations. Because the RWE is used in various fields, including quantum mechanics, optics, signal processing, and general relativity, the solutions provided here are critical for understanding a variety of intriguing physical phenomena. Moreover, the influence laws of these solutions of the Riemann wave equation cannot be overstated. Their stability, efficiency, and versatility make solitons indispensable in modern communication systems, scientific research, and engineering applications. By harnessing the power of solitons, researchers and engineers can continue to push the boundaries of wave physics and develop innovative technologies that benefit society as a whole. Furthermore, we extended some previous results such as those reported in [19]. Finally, the MATLAB software was used to demonstrate the effect of MTD on the exact solutions of the FRWE-MTD (1). We deduced that when the fractional derivative order of FRWE-MTD (1) increases, the surface moves into the right side. In the future work, we can discusses the Riemann wave equation with additive noise.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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