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About family Apostol Fubini-Euler type polynomials: Fourier expansions and integral representation

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Abstract

This paper introduces new families of Fubini-Euler type and Apostol Fubini-Euler type polynomials, providing expressions, recurrence relations, and identities. We also derive Fourier series, and integral representations, and present their rational argument representation.

Keywords: Fubini-Euler polynomials, Apostol Fubini-Euler type, series of Fourier, integral representation.

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1. Introduction

The Fourier series of a function of period T can be written in its exponential form as (see, [11, p. 19, Eq. (2.2)]):

$$f(t)=\sum_{n=-\infty}^{\infty} a_n e^{inwt}, \quad \left(w=\frac{2\pi}{T}\right),$$

where the coefficient a_n and its conjugate are computed as:

$$a_n=\frac{1}{T}\int_0^{\frac{2\pi}{w}} e^{-inwt}f(t)dt \quad \text{and} \quad a_{-n}=\frac{1}{T}\int_0^{\frac{2\pi}{w}} e^{inwt}f(t)dt.$$

The Fourier series of several families of polynomials have been introduced by various authors [4, 8, 9, 15, 16, 22], using the Lipschitz summation formula, another method used is Cauchy residue theorem. We

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begin by recalling here definitions as follows. The Fubini numbers $w_g(n)$ are defined using the following generating function (see, [12, p. 11, Eq. (2.2)]):

$$\frac{1}{2-e^z}=\sum_{n=0}^{\infty} w_g(n)\frac{z^n}{n!}, \quad |z|<\ln 2,$$

with $w_g(0)=1$. Also, the Fubini-type numbers, can be obtained from the following generating function (see [14, p. 1608, Eq. (13)]):

$$\frac{e^z-1}{2-e^z}=\sum_{n=0}^{\infty} w_M(n)\frac{z^n}{n!}, \quad |z|<\ln 2,$$

we see that $w_M(0)=0$. A new family of numbers a_n and polynomial $a_n^{(\alpha)}(x)$ was recently defined, which are obtained by making some modifications to the numbers $w_g(n)$, where a_n is given by the following generating function (see, [14, p. 1609, Eq. (14)]):

$$\frac{2}{(2-e^z)^2}=\sum_{n=0}^{\infty} a_n\frac{z^n}{n!}, \quad |z|<\ln 2.$$

The generalized Fubini type polynomials $a_n^{(\alpha)}(x)$ of order α are defined by means of the following generating function (see [14, p. 1611, Eq. (18)]):

$$\left(\frac{2}{(2-e^z)^2}\right)^{\alpha} e^{xz}=\sum_{n=0}^{\infty} a_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z|<\ln 2,$$

where $\alpha\in\mathbb{N}_0$. Observe that $a_n^{(\alpha)}(0)=a_n^{(\alpha)}$ denotes the Fubini type numbers of order α. These types of numbers are of great importance in various branches of mathematics, engineering, and physics.

In the present paper, we define new generating functions for two kinds of Fubini-Euler polynomials, we derive their explicit expressions, recurrence relations, and some identities involving those polynomials. We also show some applications that meet this family of Apostol Fubini-Euler type polynomials. On the subject of the Apostol-type polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature, for example, see [1, 3, 5, 6, 10, 13, 17–21]. The paper is organized as follows. In Section 2, we have some previous results, and important definitions, which are used in this paper. In Section 3, we define the new families of Apostol-type Fubini-Euler polynomials and their respective numbers. Finally, in Section 4, we introduce some applications of the Fourier series and integral representation of these families of polynomials in addition to their formula in rational arguments.

2. Background and previous results

Throughout this paper, we use the following standard notions: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\{0,1,2,\ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers, for the complex logarithm, we consider the principal branch.

The Laplace transform of the function t^n is given by (see, [15, p. 2198, Eq. (3.2)]):

$$\int_0^{\infty} t^n e^{-at}dt=\frac{n!}{a^{n+1}}, \quad n\in\mathbb{N}_0, \quad \Re(a)>0. \tag{2.1}$$

The Euler polynomials $E_n(x)$ in variable x are defined by means of the generating function (see, [2, p. 804, Eq. (23.1.4)]):

$$\left(\frac{2}{e^z+1}\right)e^{xz}=\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}, \quad |z|<\pi, \tag{2.2}$$

when $x = 0$, E_n denoted the so-called Euler numbers associated with the generating function.

The Fourier series of (2.2), which was obtained using the Lipschitz summation formula, is given by (see [15, p. 2197, Eq. (2.12)]):

$$E_n(x) = \frac{2}{(\pi i)^{n+1}} n! \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi i k x}}{[(2k-1)]^{n+1}}.$$

The Apostol-Euler polynomials are defined by the following generating function (see [15, p. 2194, Eq. (1.4)]):

$$\left(\frac{2}{\lambda e^z + 1}\right) e^{zx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{z^n}{n!}, \tag{2.3}$$

with $|z| < \pi$, if $\lambda = 1$, $|z| < |\log(-\lambda)|$ if $\lambda \neq 1$, for the Apostol Euler numbers $\mathcal{E}_n(\lambda)$ we readily find from (2.3) that $\mathcal{E}_n(x;1) = E_n(x)$ and $\mathcal{E}_n(1) = E_n$.

The Fourier series of (2.3), which was obtained using the Lipschitz summation formula, is given by (see, [15, p. 2196, Theorem 2.2]):

$$\mathcal{E}_n(x;\lambda) = \frac{2n!}{\lambda^x} \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi i k x}}{[(2k-1)\pi i - \log(\lambda)]^{n+1}}.$$

The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ is defined by (see [16, p. 8 Eq (4.1)]):

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{2.4}$$

$a \in \mathbb{C}\diagdown\mathbb{Z}_0^-$, $s \in \mathbb{C}$, when $|z| < 1; \Re(s) > 1$ when $|z| = 1$. For $z = 1$ in (2.4) we have Hurwitz zeta functions

$$\zeta(s, a) = \Phi(1, s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \Re(s) > 1; a \notin \mathbb{Z}_0^-.$$

Recently several authors have continued working with different families of polynomials and introducing their representation in Fourier series. In [22], they presented the Fourier series of generalized Apostol Frobenius-Euler type polynomials, using Cauchy's residue theorem. In [8] they presented the Fourier series for the higher-order Apostol-Genocchi, Apostol-Bernoulli, and Apostol-Euler polynomials using Laurent series and residue. In [9] they introduced the Fourier series expansion of Apostol-type Frobenius-Euler of complex parameters and order α using Cauchy's residue theorem.

3. New families of Fubini-Euler type polynomials and Apostol Fubini-Euler type polynomials

Definition 3.1. The new family of Fubini-Euler type polynomials $F_n(x)$ in variable x is defined by the generating function

$$\left(\frac{1-2e^z}{e^z+1}\right) e^{xz} = \sum_{n=0}^{\infty} F_n(x)\frac{z^n}{n!}, \quad |z| < \pi. \tag{3.1}$$

The first Fubini-Euler type polynomials $F_n(x)$ are

$$F_0(x) = -\frac{1}{2}, \qquad F_1(x) = -\frac{x}{2} - \frac{3}{4}, \qquad F_2(x) = -\frac{1}{2}x^2 - \frac{3}{2}x,$$
$$F_3(x) = -\frac{1}{2}x^3 - \frac{9}{4}x^2 + \frac{3}{8}, \qquad F_4(x) = -\frac{1}{2}x^4 - 3x^3 + \frac{3}{2}x, \qquad F_5(x) = -\frac{1}{2}x^5 - \frac{15}{4}x^4 + \frac{15}{4}x^2 - \frac{3}{4}.$$

For $x = 0$ in (3.1) is also obtained, the Fubini-Euler type numbers are defined by the generating function

$$\frac{1-2e^z}{e^z+1} = \sum_{n=0}^{\infty} \frac{F_n z^n}{n!}, \quad |z| < \pi. \tag{3.2}$$

Some of these numbers are

$$F_0 = -\frac{1}{2}, \quad F_1 = -\frac{3}{4}, \quad F_2 = 0, \quad F_3 = \frac{3}{8}, \quad F_4 = 0, \quad F_5 = -\frac{3}{4}.$$

A consequence of (3.1) and (3.2) is the following proposition.

Proposition 3.2. *Let $\{F_n(x)\}_{n\geqslant 0}$ be the sequences of Fubini-Euler type polynomials in the variable x. Then the following statements hold.*

a) *Summation formula: for every $n \geqslant 0$,*

$$F_n(x) = \sum_{k=0}^{n} \binom{n}{k} F_k x^{n-k}, \qquad F_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} F_k(x).$$

b) *Differential relations (Appell polynomial sequences):*

$$\frac{\partial^k F_n(x)}{\partial x^k} = \frac{n!}{(n-k)!} F_{n-k}(x).$$

c) *Integral formulas:*

$$\int_x^{x+1} F_{n-1}(t)\,dt = \frac{x^n - 2(x+1)^n - 2F_n(x)}{n}, \quad n \in \mathbb{N}.$$

d) *Formulas for connections:*

$$\begin{aligned} F_n(x) &= \frac{E_n(x) - 2E_n(x+1)}{2}, \\ F_n(x+1) &= x^n - 2(x+1)^n - F_n(x), \\ F_{n+1}(x) &= xF_n(x) - \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} E_{n-k}F_k(x+1) - E_n(x+1). \end{aligned}$$

Theorem 3.3. *The Fubini-Euler type numbers satisfy the following recurrence relationship:*

$$\sum_{k=0}^{n} \binom{n}{k} F_k = -(2+F_n), \; n \geqslant 1, \quad \textit{and} \quad F_0 = -\frac{1}{2}.$$

Proof. From (3.2), we have

$$1 - 2e^z = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + 1\right) \sum_{n=0}^{\infty} F_n \frac{z^n}{n!}.$$

Then,

$$1 - 2\sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n=0}^{\infty} F_n \frac{z^n}{n!} + \sum_{n=0}^{\infty} F_n \frac{z^n}{n!}.$$

Similarly

$$1-\sum_{n=0}^{\infty}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty}F_n\frac{z^n}{n!}+\sum_{n=0}^{\infty}F_n\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{z^n}{n!},$$
$$-\sum_{n=1}^{\infty}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty}F_n\frac{z^n}{n!}+\sum_{n=0}^{\infty}F_n\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{z^n}{n!},$$
$$-\sum_{n=1}^{\infty}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}F_k\frac{z^n}{n!}+\sum_{n=0}^{\infty}F_n\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

By matching coefficients $\frac{z^n}{n!}$, we complete the proof. □

Definition 3.4. The new family of Apostol Fubini-Euler type polynomials $F_n(x;\lambda)$ in variable x is defined by the generating function

$$\left(\frac{1-2e^z}{\lambda e^z+1}\right)e^{xz}=\sum_{n=0}^{\infty}F_n(x;\lambda)\frac{z^n}{n!},\quad |z|<|\log(-\lambda)|. \tag{3.3}$$

The Apostol Fubini-Euler type polynomials $F_n(x;\lambda)$ are

$$F_0(x;\lambda)=-\frac{1}{\lambda+1},$$
$$F_1(x;\lambda)=-\frac{(1+\lambda)x+(2+\lambda)}{(\lambda+1)^2},$$
$$F_2(x;\lambda)=-\frac{\left[\left(\lambda^2+2\lambda+1\right)x^2+\left(2\lambda^2+6\lambda+4\right)x+\left(-\lambda^2-\lambda+2\right)\right]}{(\lambda+1)^3}.$$

For $x=0$ in (3.3), the Apostol Fubini-Euler type numbers are defined by the generating function

$$\frac{1-2e^z}{\lambda e^z+1}=\sum_{n=0}^{\infty}\frac{F_n(\lambda)z^n}{n!},\quad |z|<|\log(-\lambda)|. \tag{3.4}$$

Some of these numbers are

$$F_0(\lambda)=-\frac{1}{\lambda+1},\qquad F_1(\lambda)=-\frac{2+\lambda}{(\lambda+1)^2},\qquad F_2(\lambda)=\frac{(\lambda^2+\lambda-2)}{(\lambda+1)^3},$$
$$F_3(\lambda)=-\frac{(\lambda^3-2\lambda^2-7\lambda+2)}{(\lambda+1)^4},\qquad F_4(\lambda)=-\frac{(-\lambda^4+9\lambda^3+11\lambda^2-21\lambda+2)}{(\lambda+1)^5}.$$

Theorem 3.5. *The Apostol Fubini-Euler type numbers satisfy the following recurrence relationship:*

$$\lambda\sum_{k=0}^{n}\binom{n}{k}F_k(\lambda)+F_n(\lambda)+2=0;\quad n\geqslant 1,\quad \textit{with}\quad F_0(\lambda)=-\frac{1}{\lambda+1}.$$

A consequence of (3.3) and (3.4) is the following proposition.

Proposition 3.6. *Let $\{F_n(x;\lambda)\}_{n\geqslant 0}$ be the sequences of Apostol Fubini-Euler type polynomials in the variable x. Then the following statements hold.*

a) *Summation formula: for every $n\geqslant 0$,*

$$F_n(x;\lambda)=\sum_{k=0}^{n}\binom{n}{k}F_k(\lambda)x^{n-k},\qquad F_n(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}F_k(x;\lambda).$$

b) *Differential relations (Appell polynomial sequences):*

$$\frac{\partial^k F_n(x;\lambda)}{\partial x^k} = \frac{n!}{(n-k)!} F_{n-k}(x;\lambda).$$

c) *Integral formulas:*

$$\int_x^{x+1} F_{n-1}(t,\lambda)dt = \frac{x^n - 2(x+1)^n - 2F_n(x;\lambda)}{n}, \quad n \in \mathbb{N}.$$

d) *Formulas for connections:*

$$F_n(x;\lambda) = \frac{\mathcal{E}_n(x;\lambda) - 2\mathcal{E}_n(x+1;\lambda)}{2},$$

$$F_{n+1}(x;\lambda) = xF_n(x;\lambda) - \frac{\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}(\lambda)F_k(x+1;\lambda) - \mathcal{E}_n(x+1;\lambda),$$

$$\lambda F_n(x+1;\lambda) + F_n(x;\lambda) = x^n - 2(x+1)^n.$$

Proof. The proof of Proposition 3.6 is proved by using (3.3). □

Definition 3.7. For a real or complex parameter α and λ, the generalized Fubini-Euler type and generalized Apostol Fubini-Euler type polynomials $F_n^{(\alpha)}(x)$ and $F_n^{(\alpha)}(x;\lambda)$ of degree n in variable x are defined by means of the following generating functions:

$$\left(\frac{1-2e^z}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z| < \pi, \quad 1^{\alpha} := 1, \tag{3.5}$$

$$\left(\frac{1-2e^z}{\lambda e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \quad |z| < |\log(-\lambda)|, \quad 1^{\alpha} := 1. \tag{3.6}$$

From (3.5) and (3.6), it is fairly straightforward to deduce the addition formulas:

$$F_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n}\binom{n}{k}F_k^{(\alpha)}(x)F_{n-k}^{(\beta)}(y), \tag{3.7}$$

$$F_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}F_k^{(\alpha)}(x;\lambda)F_{n-k}^{(\beta)}(y;\lambda). \tag{3.8}$$

Making an adequate substitution in (3.7) and (3.8), we get

$$F_n^{(\alpha)}(x+y) = \sum_{k=0}^{n}\binom{n}{k}F_k^{(\alpha)}(y)x^{n-k}, \quad F_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}F_k^{(\alpha)}(y;\lambda)x^{n-k}.$$

4. Fourier expansions of the Apostol Fubini-Euler type and Fubini-Euler type polynomials

This section introduces the Fourier series and representation integral of the Apostol Fubini-Euler type and Fubini-Euler type polynomials. In addition, we obtain the formula in rational arguments for these two families.

Lemma 4.1. *Let C_N be the circle centered at the origin, with radius $(2N+1+\varepsilon)\pi$, $N \in \mathbb{Z}^+$ and ε being a fixed real, such that $(\varepsilon\pi i \pm \log(\lambda) \neq 0 (\mathrm{mod}\, \pi i))$, then for $N \to \infty$, $n \geqslant 1$, and $0 \leqslant x \leqslant 1$ we have*

$$\int_{C_N} \frac{1}{z^{n+1}}\frac{1-2e^z}{\lambda e^z+1}e^{xz}dz = 0.$$

Proof.
$$\left|\int_{C_N} \frac{1}{z^{n+1}}\frac{1-2e^z}{\lambda e^z+1}e^{xz}dz\right| \leqslant \int_{C_N} \frac{1}{|z^{n+1}|}\frac{|1-2e^z|}{|\lambda e^z+1|}|e^{xz}|\,|dz|.$$
For, $0 \leqslant x \leqslant 1$, $|\lambda e^z+1| > |\lambda e^z|$, let $z = x+iy$. Then
$$\frac{|1-2e^z|}{|\lambda e^z+1|}|e^{xz}| \leqslant \frac{|1-2e^z|}{|\lambda|\,|e^z|}|e^{xz}| \leqslant \left(\frac{|1|+|2e^z|}{|\lambda|\,|e^z|}\right)|e^{xz}| \leqslant \left(\frac{1+2e^x}{|\lambda|\,e^{\Re(z)}}\right)e^{x\Re(z)} \leqslant \frac{1+2e}{|\lambda|}.$$
So that
$$\left|\int_{C_N} \frac{1}{z^{n+1}}\frac{1-2e^z}{\lambda e^z+1}e^{xz}dz\right| \leqslant \frac{1+2e}{|\lambda|}\int_{C_N}\frac{|dz|}{|z^{n+1}|} = \frac{2+4e}{|\lambda|\,((2N+1+\varepsilon)\pi)^n}.$$
As the value of $N \to \infty$, the previous expression tends to 0. Therefore, when $N \to \infty$, $n \geqslant 1$, the integral of the Lemma tends to 0, this completes the proof. □

Theorem 4.2. *Let* $\lambda \in \mathbb{C}\backslash\{0;1;-2\}, n \geqslant 1, 0 \leqslant x \leqslant 1$, *we have*
$$F_n(x,\lambda) = n!\left(\frac{1}{\lambda}\right)^x\left[\frac{\lambda+2}{\lambda}\right]\sum_{k\in\mathbb{Z}}\frac{e^{(2k-1)\pi i x}}{[(2k-1)\pi i-\log(\lambda)]^{n+1}}. \tag{4.1}$$

Proof. First we consider the function $f_n(z) = \frac{1}{z^{n+1}}\frac{1-2e^z}{\lambda e^z+1}e^{xz}$ and the integral of Lemma 4.1:
$$\int_{C_N} f_n(z)dz.$$
The poles of the function $f_n(z)$ are given by
$$z_k = 2\pi k i - \pi i - \log(\lambda), \qquad k \in \mathbb{Z}.$$
With $z = 0$ we have a pole of order $n+1$. From the Cauchy Residue theorem we have (see, [7, p. 112, Theorem 2.2]):
$$\int_{C_N} f_n(z)dz = 2\pi i\left\{\mathrm{Res}(f_n(z), z=0) + \sum_{k\in\mathbb{Z}}\mathrm{Res}(f_n(z), z=z_k)\right\}. \tag{4.2}$$
We calculate $\mathrm{Res}(f_n(z), z=0)$ and $\mathrm{Res}(f_n(z), z=z_k)$ as follows (see, [7, p. 113, Proposition 2.4])
$$\begin{aligned}\mathrm{Res}(f_n(z), z=0) &= \lim_{z\longrightarrow 0}\frac{1}{n!}\frac{d^n}{dz^n}\left[(z-0)^{n+1}\frac{1}{z^{n+1}}\sum_{j=0}^{\infty}F_j(x,\lambda)\frac{z^j}{j!}\right]\\ &= \lim_{z\longrightarrow 0}\frac{1}{n!}\sum_{j=n}^{\infty}F_j(x;\lambda)\frac{z^{j-n}}{(j-n)!} = \frac{F_n(x;\lambda)}{n!},\end{aligned}$$
and
$$\begin{aligned}\mathrm{Res}(f_n(z), z=z_k) &= \lim_{z\longrightarrow z_k}(z-z_k)(z)^{-(n+1)}\frac{1-2e^z}{\lambda e^z+1}e^{xz}\\ &= \frac{1}{z_k^{n+1}}e^{xz_k}\left[\lim_{z\longrightarrow z_k}\frac{z-z_k}{\lambda e^z+1} - \lim_{z\longrightarrow z_k}\frac{2e^z(z-z_k)}{\lambda e^z+1}\right]\\ &= \frac{1}{z_k^{n+1}}e^{xz_k}\left[\frac{1}{\lambda e^{z_k}} - \frac{2}{\lambda}\right] = \frac{e^{2\pi k x i - \pi x i - x\log(\lambda)}}{[2\pi k i - \pi i - \log(\lambda)]^{n+1}}\left[\frac{1}{\lambda e^{2\pi k i - \pi - \log(\lambda)}} - \frac{2}{\lambda}\right].\end{aligned}$$

So, in (4.2), we have

$$\int_{C_N} f_n(z)dz = 2\pi i \left\{ \frac{F_n(x;\lambda)}{n!} + \sum_{k\in\mathbb{Z}} \frac{e^{2\pi kxi-\pi xi-x\log(\lambda)}}{[2\pi ki-\pi i-\log(\lambda)]^{n+1}} \left[\frac{1}{\lambda e^{2\pi ki-\pi i-\log(\lambda)}} - \frac{2}{\lambda} \right] \right\}.$$

Taking $N \to \infty$ in Lemma 4.1, it becomes $\int_{C_N} f_n(z)dz = 0$. So we have

$$F_n(x;\lambda) = \left[\frac{\lambda+2}{\lambda}\right] \frac{n!}{(\lambda)^x} \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi ix}}{[(2k-1)\pi i-\log(\lambda)]^{n+1}},$$

this completes the proof. □

Corollary 4.3. *Letting* $\lambda = 1, n \geqslant 1, 0 \leqslant x \leqslant 1$, *we have*

$$F_n(x) = F_n(x;1) = \frac{3}{(2\pi i)^{n+1}} n! \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi ix}}{[(k-\frac{1}{2})]^{n+1}}.$$

Next, the integral representation of Apostol Fubini-Euler type polynomials and Fubini-Euler type polynomials are presented.

Theorem 4.4. *For* $n \in \mathbb{N}$ *and* $0 < x \leqslant 1$, $|\xi| < \frac{1}{2}$, $\xi \in \mathbb{R}$, *we have*

$$\begin{aligned} F_n(x;e^{2\pi i\xi}) = {} & \frac{\Theta e^{-2\xi\pi ix}}{2} \left[\int_0^\infty \frac{D(n;x,\nu)(e^{2\pi(\nu-ix)}e^{2\xi\pi\nu} + e^{-2\xi\pi\nu})}{\cosh 2\pi\nu - \cos 2\pi x} \nu^n d\nu \right] \\ & + \frac{\Theta e^{-2\xi\pi ix}}{2} \left[\int_0^\infty \frac{iB(n;x,\nu)(e^{2\pi(\nu-ix)}e^{2\xi\pi\nu} - e^{-2\xi\pi\nu})}{\cosh 2\pi\nu - \cos 2\pi x} \nu^n d\nu \right], \end{aligned}$$

where

$$D(n;x,\nu) = [e^{\pi\nu}\cos(\pi x - \frac{(n+1)\pi}{2}) - e^{-\pi\nu}\cos(\pi x + \frac{(n+1)\pi}{2})],$$
$$B(n;x,\nu) = [e^{\pi\nu}\sin(\pi x - \frac{(n+1)\pi}{2}) - e^{-\pi\nu}\sin(\pi x + \frac{(n+1)\pi}{2})].$$

Proof. As previously demonstrated, the Fourier series of the Apostol Fubini-Euler type polynomials is given by:

$$F_n(x;\lambda) = n! \frac{1}{(\lambda)^x} \left(\frac{\lambda+2}{\lambda}\right) \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi ix}}{[(2k-1)\pi i-\log(\lambda)]^{n+1}},$$

if $\lambda = e^{2\pi i\xi}$ and $k \mapsto -k$, we have

$$\begin{aligned} F_n(x;e^{2\pi i\xi}) &= n!\left(2e^{-2\pi i\xi}+1\right) \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi ix\xi}e^{(-2k-1)\pi ix}}{[-2k\pi i-\pi i-2\pi i\xi]^{n+1}} \\ &= n!\left(2e^{-2\pi i\xi}+1\right) \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi ix\xi}e^{-(2k+1)\pi ix}}{[-(2k+1)\pi i-2\pi i\xi]^{n+1}} \\ &= n!\left(2e^{-2\pi i\xi}+1\right) \sum_{k\in\mathbb{Z}} \frac{e^{-(2\xi+2k+1)\pi ix}}{[-\pi i(2k+1+2\xi)]^{n+1}} \\ &= \frac{n!}{(-\pi i)^{n+1}}\left(2e^{-2\pi i\xi}+1\right) \sum_{k\in\mathbb{Z}} \frac{e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}. \end{aligned}$$

Here $\Theta = \left(2e^{-2\pi i\xi}+1\right)$, now applying the well-known formula (2.1) we have

$$\begin{aligned}
F_n(x;e^{2\pi i\xi}) &= \frac{1}{(-\pi i)^{n+1}}\Theta\sum_{k\in\mathbb{Z}}\frac{n!e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}\frac{n!e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}+\sum_{-\infty}^{k=0}\frac{n!e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}\frac{n!e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}+\sum_{k=0}^{\infty}\frac{n!e^{-(2\xi-2k+1)\pi ix}}{[-2k+2\xi+1]^{n+1}}\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}\frac{n!e^{-(2\xi+2k+1)\pi ix}}{[2k+2\xi+1]^{n+1}}+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{n!e^{(2k-2\xi-1)\pi ix}}{[2k-2\xi-1]^{n+1}}\right\},
\end{aligned}$$

$$\begin{aligned}
F_n(x;e^{2\pi i\xi}) &= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}e^{-(2\xi+2k+1)\pi ix}\int_0^{\infty}t^ne^{-(2k+2\xi+1)t}dt\right.\\
&\quad\left.+(-1)^{n+1}\sum_{k=0}^{\infty}e^{(2k-2\xi-1)\pi ix}\int_0^{\infty}t^ne^{-(2k-2\xi-1)t}dt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}\int_0^{\infty}e^{-(2k+2\xi+1)\pi ix}e^{-(2k+2\xi+1)t}t^ndt\right.\\
&\quad\left.+(-1)^{n+1}\sum_{k=0}^{\infty}\int_0^{\infty}e^{(2k-2\xi-1)\pi ix}e^{-(2k-2\xi-1)t}t^ndt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\sum_{k=0}^{\infty}\int_0^{\infty}e^{-(2\xi+1)\pi ix}e^{-(2\xi+1)t}e^{-2(\pi ix+t)k}t^ndt\right.\\
&\quad\left.+(-1)^{n+1}\sum_{k=0}^{\infty}\int_0^{\infty}e^{-(2\xi+1)\pi ix}e^{(2\xi+1)t}e^{2(\pi ix-t)k}t^ndt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{e^{-(2\xi+1)\pi ix}\int_0^{\infty}e^{-(2\xi+1)t}t^n\sum_{k=0}^{\infty}e^{-2(\pi ix+t)k}dt\right.\\
&\quad\left.+(-1)^{n+1}e^{-(2\xi+1)\pi ix}\int_0^{\infty}e^{(2\xi+1)t}t^n\sum_{k=0}^{\infty}e^{2(\pi ix-t)k}dt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{e^{-(2\xi+1)\pi ix}\int_0^{\infty}e^{-(2\xi+1)t}t^n\frac{e^{2t}}{e^{2t}-e^{-2\pi ix}}dt\right.\\
&\quad\left.+(-1)^{n+1}e^{-(2\xi+1)\pi ix}\int_0^{\infty}e^{(2\xi+1)t}t^n\frac{e^{2t}}{e^{2t}-e^{2\pi ix}}dt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\int_0^{\infty}\frac{e^{-(2\xi+1)\pi ix}}{e^{2t}-e^{-2\pi ix}}e^{2t}e^{-(2\xi+1)t}t^ndt+(-1)^{n+1}\int_0^{\infty}\frac{e^{-(2\xi+1)\pi ix}}{e^{2t}-e^{2\pi ix}}e^{2t}e^{(2\xi+1)t}t^ndt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\int_0^{\infty}\frac{e^{-(2\xi+1)\pi ix}}{e^{2t}-e^{-2\pi ix}}e^{(2t-2\xi t-t)}t^ndt+(-1)^{n+1}\int_0^{\infty}\frac{e^{-(2\xi+1)\pi ix}}{e^{2t}-e^{2\pi ix}}e^{(2t+2\xi t+t)}t^ndt\right\}\\
&= \frac{1}{(-\pi i)^{n+1}}\Theta\left\{\int_0^{\infty}\frac{e^{-2\xi\pi ix}e^{-\pi ix}}{e^{2t}-e^{-2\pi ix}}e^{(1-2\xi)t}t^ndt+(-1)^{n+1}\int_0^{\infty}\frac{e^{-2\xi\pi ix}e^{-\pi ix}}{e^{2t}-e^{2\pi ix}}e^{(3+2\xi)t}t^ndt\right\}\\
&= \frac{\Theta e^{-2\xi\pi ix}}{(-\pi i)^{n+1}}\left\{\int_0^{\infty}\frac{e^{-\pi ix}}{e^{2t}-e^{-2\pi ix}}e^{-(2\xi-1)t}t^ndt+(-1)^{n+1}\int_0^{\infty}\frac{e^{-\pi ix}}{e^{2t}-e^{2\pi ix}}e^{(2\xi+3)t}t^ndt\right\},
\end{aligned}$$

on the other hand,

$$\frac{e^{-\pi ix}}{e^{2t}-e^{-2\pi ix}}=\frac{e^{-\pi ix}-e^{-2t}e^{\pi ix}}{e^{2t}+e^{-2t}-e^{-2\pi ix}-e^{2\pi ix}},$$

in effect, multiplying crosswise gives

$$e^{-\pi ix}e^{2t}+e^{-\pi ix}e^{-2t}-e^{-3\pi ix}-e^{\pi ix}=e^{-\pi ix}e^{2t}-e^{\pi ix}-e^{-3\pi ix}+e^{-2t}e^{-\pi ix}.$$

Analogously

$$\frac{e^{-\pi ix}}{e^{2t}-e^{2\pi ix}}=\frac{e^{-\pi ix}-e^{-2t}e^{-3\pi ix}}{e^{2t}+e^{-2t}-e^{2\pi ix}-e^{-2\pi ix}},$$

replacing in the last integral expression and taking into account that $(-\frac{1}{i})^{n+1}=e^{\frac{(n+1)\pi i}{2}}$ and $(-1)^{n+1}=e^{-(n+1)\pi i}$,

$$\begin{aligned}
F_n(x;e^{2\pi i\xi})&=\frac{\Theta e^{-2\xi\pi ix}}{(-\pi i)^{n+1}}\left\{\int_0^\infty\frac{e^{-\pi ix}-e^{-2t}e^{\pi ix}}{e^{2t}+e^{-2t}-e^{-2\pi ix}-e^{2\pi ix}}e^{-(2\xi-1)t}t^n dt\right.\\
&\qquad\left.+(-1)^{n+1}\int_0^\infty\frac{e^{-\pi ix}-e^{-2t}e^{-3\pi ix}}{e^{2t}+e^{-2t}-e^{-2\pi ix}-e^{2\pi ix}}e^{(2\xi+3)t}t^n dt\right\}\\
&=\frac{\Theta e^{-2\xi\pi ix}}{(-\pi i)^{n+1}}\left\{\int_0^\infty\frac{(e^{-2\pi ix}-e^{-2t})e^{\pi ix}}{[2\cosh 2t-2\cos 2\pi]}e^{-(2\xi-1)t}t^n dt\right.\\
&\qquad\left.+(-1)^{n+1}\int_0^\infty\frac{(e^{2\pi ix}-e^{-2t})e^{-3\pi ix}}{[2\cosh 2t-2\cos 2\pi x]}e^{(2\xi+3)t}t^n dt\right\}\\
&=\frac{\Theta e^{-2\xi\pi ix}}{2\pi^{n+1}}\left\{\int_0^\infty\frac{e^{\frac{(n+1)\pi i}{2}}(e^{-2\pi ix}-e^{-2t})e^{\pi ix}}{[\cosh 2t-\cos 2\pi x]}e^{-(2\xi-1)t}t^n dt\right.\\
&\qquad\left.+\int_0^\infty\frac{e^{-(n+1)\pi i}e^{\frac{(n+1)\pi i}{2}}(e^{2\pi ix}-e^{-2t})e^{-3\pi ix}}{[\cosh 2t-\cos 2\pi x]}e^{(2\xi+3)t}t^n dt\right\}\\
&=\frac{\Theta e^{-2\xi\pi ix}}{2\pi^{n+1}}\left\{\int_0^\infty\frac{e^{\frac{(n+1)\pi i}{2}}(e^{-2\pi ix}-e^{-2t})e^{\pi ix}}{[\cosh 2t-\cos 2\pi x]}e^{-(2\xi-1)t}t^n dt\right.\\
&\qquad\left.+\int_0^\infty\frac{e^{-\frac{(n+1)\pi i}{2}}(e^{2\pi ix}-e^{-2t})e^{-3\pi ix}}{[\cosh 2t-\cos 2\pi x]}e^{(2\xi+3)t}t^n dt\right\}.
\end{aligned}$$

Now by making $t=\pi v$,

$$\begin{aligned}
F_n(x;e^{2\pi i\xi})&=\frac{\Theta e^{-2\xi\pi ix}}{2\pi^{n+1}}\left\{\int_0^\infty\frac{e^{\frac{(n+1)\pi i}{2}}(e^{-2\pi ix}-e^{-2\pi v})e^{\pi ix}}{[\cosh 2\pi v-\cos 2\pi x]}e^{-(2\xi-1)\pi v}(\pi v)^n\pi dv\right.\\
&\qquad\left.+\int_0^\infty\frac{e^{-\frac{(n+1)\pi i}{2}}(e^{2\pi ix}-e^{-2\pi v})e^{-3\pi ix}}{[\cosh 2\pi v-\cos 2\pi x]}e^{(2\xi+3)\pi v}(\pi v)^n\pi dv\right\}\\
&=\frac{\Theta e^{-2\xi\pi ix}}{2}\left\{\int_0^\infty\frac{e^{\frac{(n+1)\pi i}{2}}(e^{-2\pi ix}-e^{-2\pi v})e^{\pi ix}}{[\cosh 2\pi v-\cos 2\pi x]}e^{-(2\xi-1)\pi v}v^n du\right.\\
&\qquad\left.+\int_0^\infty\frac{e^{-\frac{(n+1)\pi i}{2}}(e^{2\pi ix}-e^{-2\pi v})e^{-3\pi ix}}{[\cosh 2\pi v-\cos 2\pi x]}e^{(2\xi+3)\pi v}v^n dv\right\}\\
&=\frac{\Theta e^{-2\xi\pi ix}}{2}\left\{\int_0^\infty\frac{e^{\frac{(n+1)\pi i}{2}}(e^{-2\pi ix}-e^{-2\pi v})e^{\pi ix}}{[\cosh 2\pi v-\cos 2\pi x]}e^{-2\xi\pi v}e^{\pi v}v^n dv\right.
\end{aligned}$$

$$+\int_0^\infty \frac{e^{-\frac{(n+1)\pi i}{2}}(e^{2\pi ix}-e^{-2\pi\nu})e^{-3\pi ix}}{[\cosh 2\pi\nu-\cos 2\pi x]}e^{2\xi\pi\nu}e^{3\pi\nu}\nu^n d\nu\Bigg\}$$

$$=\frac{\Theta e^{-2\xi\pi ix}}{2}\Bigg\{\int_0^\infty \frac{e^{\frac{(n+1)\pi i}{2}}(e^{\pi\nu}e^{-2\pi ix}-e^{-\pi\nu})e^{\pi ix}}{[\cosh 2\pi\nu-\cos 2\pi x]}e^{-2\xi\pi\nu}\nu^n d\nu$$

$$+\int_0^\infty \frac{e^{-\frac{(n+1)\pi i}{2}}(e^{3\pi\nu}e^{2\pi ix}-e^{\pi u})e^{-3\pi ix}}{[\cosh 2\pi\nu-\cos 2\pi x]}e^{2\xi\pi\nu}\nu^n d\nu\Bigg\}$$

$$=\frac{\Theta e^{-2\xi\pi ix}}{2}\Bigg\{\int_0^\infty \frac{e^{\frac{(n+1)\pi i}{2}}(e^{\pi\nu}e^{-\pi ix}-e^{-\pi\nu}e^{\pi ix})}{[\cosh 2\pi\nu-\cos 2\pi x]}e^{-2\xi\pi\nu}\nu^n d\nu$$

$$+\int_0^\infty \frac{e^{-\frac{(n+1)\pi i}{2}}(e^{3\pi\nu}e^{-\pi ix}-e^{\pi\nu}e^{-3\pi ix})}{[\cosh 2\pi\nu-\cos 2\pi x]}e^{2\xi\pi\nu}\nu^n d\nu\Bigg\}$$

$$=\frac{\Theta e^{-2\xi\pi ix}}{2}\Bigg\{\int_0^\infty \frac{e^{-2\xi\pi\nu}(e^{\pi\nu}e^{-\pi ix}e^{\frac{(n+1)\pi i}{2}}-e^{-\pi\nu}e^{\pi ix}e^{\frac{(n+1)\pi i}{2}})}{[\cosh 2\pi\nu-\cos 2\pi x]}\nu^n d\nu$$

$$+\int_0^\infty \frac{e^{2\xi\pi\nu}(e^{3\pi\nu}e^{-\pi ix}e^{-\frac{(n+1)\pi i}{2}}-e^{\pi\nu}e^{-3\pi ix}e^{-\frac{(n+1)\pi i}{2}})}{[\cosh 2\pi\nu-\cos 2\pi x]}\nu^n d\nu\Bigg\}$$

$$=\frac{\Theta e^{-2\xi\pi ix}}{2}\Bigg\{\int_0^\infty \frac{e^{-2\xi\pi\nu}(e^{\pi\nu}e^{-(\pi x-\frac{(n+1)\pi}{2})i}-e^{-\pi\nu}e^{(\pi x+\frac{(n+1)\pi}{2})i}}{[\cosh 2\pi\nu-\cos 2\pi x]}\nu^n d\nu$$

$$+\int_0^\infty \frac{e^{2\xi\pi\nu}(e^{3\pi\nu}e^{-2\pi ix}e^{(\pi x-\frac{(n+1)\pi}{2})i}-e^{\pi\nu}e^{-2\pi ix}e^{-(\pi x+\frac{(n+1)\pi}{2})i})}{[\cosh 2\pi\nu-\cos 2\pi x]}\nu^n d\nu\Bigg\}.$$

Rearranging the integrals we have

$$F_n(x;e^{2\pi i\xi})=\frac{\Theta e^{-2\xi\pi ix}}{2}\left[\int_0^\infty \frac{D(n;x,\nu)(e^{2\pi(\nu-ix)}e^{2\xi\pi\nu}+e^{-2\xi\pi\nu})}{\cosh 2\pi\nu-\cos 2\pi x}\nu^n d\nu\right]$$
$$+\frac{\Theta e^{-2\xi\pi ix}}{2}\left[\int_0^\infty \frac{iB(n;x,\nu)(e^{2\pi(\nu-ix)}e^{2\xi\pi\nu}-e^{-2\xi\pi\nu})}{\cosh 2\pi\nu-\cos 2\pi x}\nu^n d\nu\right],$$

where

$$D(n;x,\nu)=[e^{\pi\nu}\cos(\pi x-\frac{(n+1)\pi}{2})-e^{-\pi\nu}\cos(\pi x+\frac{(n+1)\pi}{2})],$$
$$B(n;x,\nu)=[e^{\pi\nu}\sin(\pi x-\frac{(n+1)\pi}{2})-e^{-\pi\nu}\sin(\pi x+\frac{(n+1)\pi}{2})].$$

This completes the proof. □

Corollary 4.5. *For* $n\in\mathbb{N}$ *and* $0<x\leqslant 1$, *we have*

$$F_n(x)=\frac{3}{2}\int_0^\infty \frac{D(n;x,\nu)(e^{2\pi(\nu-ix)}+1)+iB(n;x,\nu)(e^{2\pi(\nu-ix)}-1)}{\cosh 2\pi\nu-\cos 2\pi x}\nu^n d\nu.$$

Proof. If in the Theorem 4.5 we let $\xi=0$, the result is obtained. □

In this part, we prove the explicit formulas for the Apostol Fubini-Euler type polynomials and Fubini-Euler type polynomials at rational arguments.

Theorem 4.6. *For $n, q \in \mathbb{N}$, $p, \xi \in \mathbb{Z}$, $|\xi| < 1$, the following expression of Apostol Fubini-Euler type polynomials at rational arguments is obtained:*

$$\begin{aligned} F_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) &= \frac{n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q} \zeta(n+1, \frac{2j+2\xi+1}{2q})e^{\left(\frac{(n+1)}{2}-\frac{(2j+2\xi+1)p}{q}\right)\pi i}\right. \\ &\quad \left.+\sum_{j=1}^{q} \zeta(n+1, \frac{2j-2\xi-3}{2q})e^{\left(\frac{-(n+1)}{2}+\frac{(2j-2\xi-3)p}{q}\right)\pi i}\right\} \\ &\quad +\frac{2n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q} \zeta(n+1, \frac{2j+2\xi+1}{2q})e^{\left(\frac{(n+1)}{2}-2\xi-\frac{(2j+2\xi+1)p}{q}\right)\pi i}\right. \\ &\quad \left.+\sum_{j=1}^{q} \zeta(n+1, \frac{2j-2\xi-3}{2q})e^{\left(\frac{(n+1)}{2}-2\xi+\frac{(2j-2\xi-3)p}{q}\right)\pi i}\right\}. \end{aligned}$$

Proof. Making some modifications to the series obtained in (4.1) and knowing $i^{n+1} = e^{\frac{(n+1)\pi i}{2}}$, we get that

$$\begin{aligned} F_n(x;\lambda) &= n!\left(\frac{\lambda+2}{\lambda}\right)\left[\frac{i^{n+1}}{\lambda^x}\right]\left[\sum_{k=0}^{\infty}\frac{e^{\left(-(\frac{n+1}{2})\pi+(2k-1)\pi x\right)i}}{[(2k-1)\pi i-\log(\lambda)]^{n+1}}\right] \\ &\quad + n!\left(\frac{\lambda+2}{\lambda}\right)\left[\frac{i^{n+1}}{\lambda^x}\right]\left[\sum_{k=1}^{\infty}\frac{e^{\left((\frac{n+1}{2})\pi-(2k+1)\pi x\right)i}}{[(2k+1)\pi i+\log(\lambda)]^{n+1}}\right]. \end{aligned} \tag{4.3}$$

The result shown below is equivalent to (4.3):

$$\begin{aligned} F_n(x;\lambda) &= n!\left(\frac{\lambda+2}{\lambda}\right)\left[\frac{i^{n+1}}{\lambda^x}\right]\left[\sum_{k=1}^{\infty}\frac{e^{\left(-(\frac{n+1}{2})\pi+(2k-3)\pi x\right)i}}{[(2k-3)\pi i-\log(\lambda)]^{n+1}}\right] \\ &\quad + n!\left(\frac{\lambda+2}{\lambda}\right)\left[\frac{i^{n+1}}{\lambda^x}\right]\left[\sum_{k=1}^{\infty}\frac{e^{\left((\frac{n+1}{2})\pi-(2k+1)\pi x\right)i}}{[(2k+1)\pi i+\log(\lambda)]^{n+1}}\right]. \end{aligned}$$

So that, in view of (2.4) and the elementary series identity (see, [15, p. 2202, eq 4.12])

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{l}\sum_{k=0}^{\infty} f(lk+j), \quad l \in \mathbb{N},$$

we find the formula:

$$\begin{aligned} F_n(x;\lambda) &= \left[\frac{\lambda+2}{\lambda}\right]\frac{n!}{(2\pi l)^{n+1}}\left\{\sum_{j=1}^{l}\Phi(e^{2l\pi x i}, n+1, \frac{(2j-3)\pi i-\log(\lambda)}{2\pi i l})e^{\left(-\frac{(n+1)\pi}{2}-3\pi x+2j\pi x\right)i}\right. \\ &\quad \left.+\sum_{j=1}^{l}\Phi(e^{-2l\pi x i}, n+1, \frac{(2j+1)\pi i+\log(\lambda)}{2\pi i l})e^{\left(\frac{(n+1)\pi}{2}-\pi x-2j\pi x\right)i}\right\}. \end{aligned} \tag{4.4}$$

Setting $\lambda = e^{2\pi i\xi}$, $x = \dfrac{p}{q}$, $l = q$ in (4.4), the result of Theorem 4.6 is obtainded and this completes the proof. □

5. Conclusion

This study unveils groundbreaking families of Fubini-Euler type polynomials and Apostol Fubini-Euler type polynomials, elucidating explicit expressions, recurrence relations, and intricate identities. The meticulously derived Fourier series and integral representations deepen the comprehension of these polynomial families. The succinct rational argument representation contributes to a refined mathematical framework.

For future exploration, potential research directions may delve into the diverse applications of these polynomials across mathematical domains, amplifying their versatility and significance. The distinctive advantage of this study lies in its substantial contribution to the evolving realm of special polynomials, offering a valuable resource for mathematicians and researchers.

Summarily, the pivotal arguments and findings underscore the profound intricacies of these polynomial families and their far-reaching implications. This research not only enriches our understanding but also catalyzes advancing mathematical discourse, unveiling novel prospects for exploration and application.

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