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Iterative Hille-type oscillation criteria of half-linear advanced dynamic equations of second order

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In this paper, we establish iterative Hille-type criteria for advanced functional half-linear dynamic equations of the second order. These results extend and improve recent criteria established by multiple authors for the same equation and encompass classical criteria. We provide an example to demonstrate the significance of the results obtained.

KEYWORDS
advanced dynamic equations, half-linear, Hille-type, oscillation, second order

MSC CLASSIFICATION
39A10, 39A21, 39A99, 34K11, 34C10, 34N05

1 | INTRODUCTION

Hilger [1] proposed the theory of dynamic equations on time scales to unify continuous and discrete analysis. Various time scales may be employed in a variety of applications. The theory of dynamic equations consists of classical theories for differential equations and difference equations, as well as other cases that fall between these classical cases. The $q-$difference equations, which have significant applications in quantum theory (see Kac & Chueng [2]), can be considered when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^{\lambda} : \lambda \in \mathbb{N}_0 \text{ for } q > 1\}$, as well as other time scales, such $\mathbb{T} = h\mathbb{N}$, $\mathbb{T}= \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ is the set of the harmonic numbers. See earlier studies [3–5] for additional details on the calculus of time scales.

Advanced dynamic equations have been established based on various practical domains where the rates of change are influenced by both present and future conditions. To incorporate the impact of potential future variables on the decision-making process, it is necessary to introduce a sophisticated term into the equation. Disciplines such as population dynamics, economic issues, and mechanical control engineering are commonly characterized by the influence of future factors on the growth of the dynamical component. We refer the reader to sources [6–9] for further details.

Oscillation has garnered significant attention among researchers in applied fields due to its origins in mechanical vibrations and its extensive applications in the realms of science and engineering. In order to denote the dependence of solutions on temporal contexts, oscillation models occasionally incorporate delays or advanced terms. Extensive research has been conducted on the topic of oscillation in delay equations, as shown by the works of earlier studies [10–21]. However, there is a scarcity of studies focusing on advanced oscillation, as indicated by the limited number of works such as [22–27].

Various models in real-world applications involve oscillation phenomena. For instance, in the field of mathematical biology, certain models incorporate oscillation and/or delay effects through the utilization of cross-diffusion terms. For further exploration of this topic, please refer to the papers [28–35]. This work encompasses the examination of differential equations due to their relevance in addressing various real-world phenomena, such as non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous media. For additional details, those interested may refer to the papers [36–44]. As a result, we study advanced dynamic equations of second-order half-linear form

$$\left[r(\xi)\,\varphi(y^{\Delta}(\xi))\right]^{\Delta} + q(\xi)\,\varphi(y(g(\xi))) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ which is assumed to be unbounded above, where $\xi \in [\xi_0, \infty)_{\mathbb{T}} := [\xi_0, \infty) \cap \mathbb{T}$, $\xi_0 \geq 0$, $\xi_0 \in \mathbb{T}$, , $\varphi(u) := |u|^{\gamma-1}u$, $\gamma > 0$, $r, q : \mathbb{T} \to (0, \infty)$ are rd-continuous functions such that

$$R(\xi) := \int_{\xi_0}^{\xi} \frac{\Delta t}{r^{1/\gamma}(t)} \to \infty \text{ as } \xi \to \infty,$$

and $g : \mathbb{T} \to \mathbb{T}$ is an rd-continuous nondecreasing function satisfying $g(\xi) \geq \xi$ on $[\xi_0, \infty)_{\mathbb{T}}$ and $\lim_{\xi\to\infty} g(\xi) = \infty$.

By a solution of Equation (1.1), we mean a nontrivial real-valued function $y \in \mathrm{C}^1_{\mathrm{rd}}[T_y, \infty)_{\mathbb{T}}$, $T_y \in [\xi_0, \infty)_{\mathbb{T}}$ such that $r\varphi(y^{\Delta})(\xi) \in \mathrm{C}^1_{\mathrm{rd}}[T_y, \infty)_{\mathbb{T}}$ and y satisfies (1.1) on $[T_y, \infty)_{\mathbb{T}}$, where C_{rd} is the set of rd-continuous functions.

If a solution y of (1.1) is neither eventually positive nor eventually negative, we call it oscillatory; otherwise, we refer to it as nonoscillatory. The solutions that vanish in some infinity will be excluded from consideration. It is said that (1.1) is oscillatory if all of its solutions are oscillatory.

We start by reviewing oscillation results for differential equations related to Equation (1.1). Hille [45] examined the oscillatory behavior of solutions of the ordinary differential equation

$$y''(\xi) + q(\xi)y(\xi) = 0 \tag{1.2}$$

and proved that if

$$\liminf_{\xi\to\infty} \xi \int_{\xi}^{\infty} q(t)\mathrm{d}t > \frac{1}{4}, \tag{1.3}$$

then Equation (1.2) is oscillatory. Erbe [46] generalized the Hille-type condition (1.3) to the delay differential equation

$$y''(\xi) + q(\xi)y(g(\xi)) = 0, \tag{1.4}$$

where $g(\xi) \leq \xi$ and showed that if

$$\liminf_{\xi\to\infty} \xi \int_{\xi}^{\infty} \frac{g(t)}{t} q(t)\mathrm{d}t > \frac{1}{4}, \tag{1.5}$$

then Equation (1.4) is oscillatory. For the case

$$\int_{\xi_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty,$$

the Hille-type criterion for the Sturm–Liouville linear equation

$$\left(r(\xi)\,y'(\xi)\right)' + q(\xi)\,y(\xi) = 0 \tag{1.6}$$

has been established and proved that if

$$\liminf_{\xi\to\infty}\left\{\int_{\xi_0}^{\xi}\frac{\mathrm{d}t}{r(t)}\int_{\xi}^{\infty}q(t)\,\mathrm{d}t\right\}>\frac{1}{4}, \tag{1.7}$$

then Equation (1.6) is oscillatory; see, for example, Agarval et al. [47, Chap. 2]. These criteria have been extended to the ordinary half-linear differential equation

$$\left(r(\xi)\,\varphi(y'(\xi))\right)'+q(\xi)\,\varphi(y(\xi))=0 \tag{1.8}$$

and obtained that if

$$\int_{\xi_0}^{\infty}\frac{\mathrm{d}t}{r^{1/\gamma}(t)}=\infty$$

and

$$\liminf_{\xi\to\infty}\left\{\left(\int_{\xi_0}^{\xi}\frac{\mathrm{d}t}{r^{1/\gamma}(t)}\right)^{\gamma}\int_{\xi}^{\infty}q(t)\,\mathrm{d}t\right\}>\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{1.9}$$

then Equation (1.8) is oscillatory; see Došlý and Řehák [48, Sec 3.1.1]. For further Hille-type criteria of some advanced differential equations that are related to Equation (1.1), see the papers [49–51].

In relation to second-order advanced dynamic equations, previous works [52–59] established several Hille-type oscillation criteria for different forms of second-order dynamic equations under some various restrictive conditions, which ensure that the solutions are oscillatory; see also previous research [Conclusions Section ,58, 59] for a good comparison between these results. Some of these results are as follows:

Theorem 1.1 (see Hassan et al. [58]). *Equation (1.1) is oscillatory if*

(1) $0<\gamma\le 1$, $g(\xi)\ge\sigma(\xi)$, *and*

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}q(t)\,\Delta t\right\}>\frac{\gamma^{\gamma}}{l^{\gamma(1-\gamma)}(\gamma+1)^{\gamma+1}}, \tag{1.10}$$

(2) $\gamma\ge 1$, $g(\xi)\ge\xi$, *and*

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}q(t)\,\Delta t\right\}>\frac{\gamma^{\gamma}}{l^{\gamma(\gamma-1)}(\gamma+1)^{\gamma+1}}, \tag{1.11}$$

where $l:=\liminf_{\xi\to\infty}\frac{R(\xi)}{R(\sigma(\xi))}>0$.

All of the results mentioned [52–59] have in common that the advanced argument $g(\xi)$ is not included in the Hille-type criteria mentioned (1.10) and (1.11); these criteria fit better with the ordinary dynamic equation

$$\left[r(\xi)\,\varphi(y^{\Delta}(\xi))\right]^{\Delta}+q(\xi)\,\varphi(y(\xi))=0$$

and do not disclose how the oscillation is dependent on the advanced argument. Moreover, these works did not discuss the cases of

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}q(t)\,\Delta t\right\}\le\frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}}. \tag{1.12}$$

This means that, more specifically, Theorem 1.1 fails to work if condition (1.12) holds.

The goal of this paper is to come up with new Hille-type criteria that can be easily and practically applied to the advanced dynamic Equation (1.1) , including a role for the advanced function $g(\xi)$, and that are also workable when the condition (1.12) is satisfied. The reader is pointed to papers about Hille-type criteria [60–64] and the sources listed therein.

2 | MAIN RESULTS

In this section, the key findings will be demonstrated and without loss of generality, all improper integrals are assumed to be convergent. If not, we find that Equation (1.1) is oscillatory, as shown in Fite [65]. This section begins with the following preliminary lemmas.

Lemma 2.1 (see Erbe et al. [66]). *Suppose $y(\xi)$ is an eventually positive solution of Equation (1.1). Then*

$$y^{\Delta}(\xi) > 0 \text{ and } \left[r(\xi)\varphi\left(y^{\Delta}(\xi)\right)\right]^{\Delta} < 0 \tag{2.1}$$

eventually.

In the following, we introduce a sequence $\{\beta_n\}_{n\in\mathbb{N}_0}$ defined by

$$\beta_{n+1} := \liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}\left(\frac{R(g(t))}{R(t)}\right)^{\gamma\sqrt[\gamma]{\beta_n}} q(t)\Delta t\right\},\ n\in\mathbb{N}_0, \tag{2.2}$$

with $\beta_0 = 0$. It is important to note that the sequence $\{\beta_n\}_{n\in\mathbb{N}_0}$ is nondecreasing. The following lemma improves and extends Lemma 2.1.

Lemma 2.2. *If $y(\xi)$ is an eventually positive solution of Equation (1.1) , then for any $n\in\mathbb{N}_0$,*

$$\left(\frac{y(\xi)}{R^{\sqrt[\gamma]{\beta_n}}(\xi)}\right)^{\Delta} > 0 \tag{2.3}$$

eventually.

Proof. Without loss of generality, let $y(\xi) > 0$ on $[\xi_0,\infty)_{\mathbb{T}}$, and according to Lemma 2.1, $y^{\Delta}(\xi) > 0$ and $\left[r(\xi)\varphi(y^{\Delta}(\xi))\right]^{\Delta} < 0$ on $[\xi_0,\infty)_{\mathbb{T}}$. Therefore, (2.3) holds if there is an $n\in\mathbb{N}_0$ such that $\beta_n = 0$. Hence, in the following, we assume that $\beta_1 > 0$, so $\beta_n > 0$ for any $n\in\mathbb{N}$ since $\{\beta_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence. Now, we show by mathematical induction that for arbitrary but fixed $k_n\in(0,1)$

$$\left(\frac{y(\xi)}{R^{k_n\sqrt[\gamma]{\beta_n}}(\xi)}\right)^{\Delta} \geq 0 \tag{2.4}$$

eventually. By using the fact that $\left[r(\xi)\varphi\left(y^{\Delta}(\xi)\right)\right]^{\Delta} < 0$ on $[\xi_0,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} y(\xi) > y(\xi)-y(\xi_0) &= \int_{\xi_0}^{\xi}\frac{\varphi^{-1}\left[r(t)\varphi\left(y^{\Delta}(t)\right)\right]}{r^{1/\gamma}(t)}\Delta t \\ &> \varphi^{-1}\left[r(\xi)\varphi\left(y^{\Delta}(\xi)\right)\right]\int_{\xi_0}^{\xi}\frac{\Delta t}{r^{1/\gamma}(t)} \\ &= \varphi^{-1}\left[r(\xi)\varphi\left(y^{\Delta}(\xi)\right)\right]R(\xi); \end{aligned}$$

that is,

$$R(\xi)\,r^{1/\gamma}(\xi)\,y^{\Delta}(\xi) < y(\xi). \tag{2.5}$$

Since $y^{\Delta}(\xi) > 0$ on $[\xi_0, \infty)_{\mathbb{T}}$, from Equation (1.1), we have that

$$\begin{aligned} r(\xi)\varphi(y^{\Delta}(\xi)) &\geq \int_{\xi}^{\infty} q(t)\varphi(y(g(t)))\Delta t \\ &\geq \int_{\xi}^{\infty} q(t)\varphi(y(t))\Delta t \\ &\geq \varphi(y(\xi))\int_{\xi}^{\infty} q(t)\,\Delta t. \end{aligned} \tag{2.6}$$

Let $\varepsilon_0 \in (0,1)$ be arbitrary but fixed. It follows from the definition (2.2) of β_1 that

$$\int_{\xi}^{\infty} q(t)\,\Delta t \geq \frac{\varepsilon_0 \beta_1}{R^{\gamma}(\xi)} \tag{2.7}$$

for sufficiently large ξ. Substituting (2.7) into (2.6), we get

$$R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) \geq k_1 \sqrt[\gamma]{\beta_1}\, y(\xi), \tag{2.8}$$

where $k_1 := \sqrt[\gamma]{\varepsilon_0} \in (0,1)$. By the quotient rule, we get

$$\left(\frac{y(\xi)}{R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)}\right)^{\Delta} = \frac{R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)\, y^{\Delta}(\xi) - \left(R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)\right)^{\Delta} y(\xi)}{R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)\, R^{k_1\sqrt[\gamma]{\beta_1}}(\sigma(\xi))}. \tag{2.9}$$

From (2.5) and (2.8), we have $k_1\sqrt[\gamma]{\beta_1} < 1$ and then from the Pötzsche chain rule, we arrive at

$$\begin{aligned} \left(R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)\right)^{\Delta} &= \frac{k_1\sqrt[\gamma]{\beta_1}}{r^{1/\gamma}(\xi)} \int_0^1 \left[(1-h)R(\xi) + hR(\sigma(\xi))\right]^{k_1\sqrt[\gamma]{\beta_1}-1} dh \\ &\leq k_1\sqrt[\gamma]{\beta_1}\,\frac{R^{k_1\sqrt[\gamma]{\beta_1}-1}(\xi)}{r^{1/\gamma}(\xi)}. \end{aligned} \tag{2.10}$$

Using (2.10) in (2.9) , we have

$$\left(\frac{y(\xi)}{R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)}\right)^{\Delta} \geq \frac{R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) - k_1\sqrt[\gamma]{\beta_1}\, y(\xi)}{r^{1/\gamma}(\xi)\, R(\xi)\, R^{k_1\sqrt[\gamma]{\beta_1}}(\sigma(\xi))}.$$

By means of (2.8), we find that

$$\left(\frac{y(\xi)}{R^{k_1\sqrt[\gamma]{\beta_1}}(\xi)}\right)^{\Delta} \geq 0.$$

Then (2.4) holds for $n = 1$. Assume that (2.4) holds for some $n \in \{0, 1, \dots, m\}$. Then

$$\left(\frac{y(\xi)}{R^{k_n\sqrt[\gamma]{\beta_n}}(\xi)}\right)^{\Delta} \geq 0 \text{ eventually.}$$

This together with (1.1) shows that

$$\begin{aligned} r(\xi)\,\varphi(y^{\Delta}(\xi)) &\geq \int_{\xi}^{\infty} q(t)\,\varphi(y(g(t)))\Delta t \\ &\geq \int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma k_n \sqrt[\gamma]{\beta_n}} \varphi(y(t))q(t)\,\Delta t \\ &\geq \varphi(y(\xi)) \int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma k_n \sqrt[\gamma]{\beta_n}} q(t)\,\Delta t. \end{aligned}$$

Let $0 < \varepsilon, \varepsilon_m < 1$ be arbitrary but fixed. It follows from definition (2.2) of β_{n+1} that for sufficiently large ξ,

$$\int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t)\,\Delta t \geq \frac{\varepsilon_n \beta_{n+1}}{R^{\gamma}(\xi)}$$

and

$$\frac{R(t)}{R(g(t))} \geq \varepsilon\lambda,$$

where

$$\lambda := \liminf_{\xi\to\infty} \frac{R(\xi)}{R(g(\xi))},\ \ 0 \leq \lambda \leq 1.$$

Therefore,

$$\begin{aligned} r(\xi)\,\varphi(y^{\Delta}(\xi)) &\geq (\varepsilon\lambda)^{\gamma \sqrt[\gamma]{\beta_n}(1-k_n)} \varphi(y(\xi)) \int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t)\,\Delta t \\ &\geq \varepsilon_n (\varepsilon\lambda)^{\gamma \sqrt[\gamma]{\beta_n}(1-k_n)} \beta_{n+1} \left(\frac{y(\xi)}{R(\xi)}\right)^{\gamma}, \end{aligned}$$

which implies that

$$\begin{aligned} R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) &\geq \sqrt[\gamma]{\varepsilon_n} (\varepsilon\lambda)^{\sqrt[\gamma]{\beta_n}(1-k_n)} \sqrt[\gamma]{\beta_{n+1}}\, y(\xi) \\ &= k_{n+1} \sqrt[\gamma]{\beta_{n+1}}\, y(\xi), \end{aligned} \tag{2.11}$$

where k_{n+1} satisfies

$$0 < k_{n+1} := \sqrt[\gamma]{\varepsilon_n} (\varepsilon\lambda)^{\sqrt[\gamma]{\beta_n}(1-k_n)} < 1.$$

Hence,

$$\left(\frac{y(\xi)}{R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)}\right)^{\Delta} = \frac{R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)\, y^{\Delta}(\xi) - \left(R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)\right)^{\Delta} y(\xi)}{R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)\, R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\sigma(\xi))}. \tag{2.12}$$

Again, from (2.5) and (2.11), we have $k_{n+1}\sqrt[\gamma]{\beta_{n+1}} < 1$ and then by the Pötzsche chain rule, we obtain

$$\left(R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)\right)^{\Delta} \leq k_{n+1}\sqrt[\gamma]{\beta_{n+1}} \frac{R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}-1}(\xi)}{r^{1/\gamma}(\xi)}. \tag{2.13}$$

Using (2.13) in (2.12) , we have

$$\left(\frac{y(\xi)}{R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\xi)}\right)^{\Delta} \geq \frac{R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) - k_{n+1}\sqrt[\gamma]{\beta_{n+1}}\, y(\xi)}{r^{1/\gamma}(\xi)\, R(\xi)\, R^{k_{n+1}\sqrt[\gamma]{\beta_{n+1}}}(\sigma(\xi))} \geq 0.$$

This demonstrates that (2.4) holds for $n+1$. Hence, (2.4) holds for all $n \in \mathbb{N}_0$. From (2.2) and (2.11) , there is a $\kappa \geq 1$ such that

$$R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) \geq k_{n+1}\sqrt[\gamma]{\beta_{n+1}}\, y(\xi) \geq k_{n+1}\kappa^{\left(\sqrt[\gamma]{\beta_n}-\sqrt[\gamma]{\beta_{n-1}}\right)}\sqrt[\gamma]{\beta_n}\, y(\xi).$$

Since $0 < k_{n+1} < 1$ is arbitrary, we can take $k_{n+1} > 1/\kappa^{\left(\sqrt[\gamma]{\beta_n}-\sqrt[\gamma]{\beta_{n-1}}\right)}$. Thus,

$$R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) > \sqrt[\gamma]{\beta_n}\, y(\xi).$$

Thus,

$$\begin{aligned}\left(\frac{y(\xi)}{R^{\sqrt[\gamma]{\beta_n}}(\xi)}\right)^{\Delta} &= \frac{R^{\sqrt[\gamma]{\beta_n}}(\xi)\, y^{\Delta}(\xi) - \left(R^{\sqrt[\gamma]{\beta_n}}(\xi)\right)^{\Delta} y(\xi)}{R^{\sqrt[\gamma]{\beta_n}}(\xi)\, R^{\sqrt[\gamma]{\beta_n}}(\sigma(\xi))}\\ &\geq \frac{R(\xi)\, r^{1/\gamma}(\xi)\, y^{\Delta}(\xi) - \sqrt[\gamma]{\beta_n}\, y(\xi)}{r^{1/\gamma}(\xi)\, R(\xi)\, R^{\sqrt[\gamma]{\beta_n}}(\sigma(\xi))} > 0.\end{aligned}$$

This illustrates that (2.3) holds for $n \in \mathbb{N}_0$. □

Theorem 2.1. *If $l > 0$ and there exists $n \in \mathbb{N}_0$ such that*

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}\left(\frac{R(g(t))}{R(t)}\right)^{\gamma\sqrt[\gamma]{\beta_n}} q(t)\,\Delta t\right\} > \frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}}, \tag{2.14}$$

where $l := \liminf_{\xi\to\infty}\frac{R(\xi)}{R(\sigma(\xi))}$, then Equation (1.1) is oscillatory.

Proof. Assume y is a nonoscillatory solution of (1.1) on $[\xi_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, let $y(\xi) > 0$ on $[\xi_0,\infty)_{\mathbb{T}}$. According to Lemma 2.2, there exists a $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that $\frac{y(\xi)}{R^{\sqrt[\gamma]{\beta_n}}(\xi)}$ is strictly increasing on $[\xi_1,\infty)_{\mathbb{T}}$ for any $n \in \mathbb{N}_0$. Define

$$x(\xi) := \frac{r(\xi)\varphi(y^{\Delta}(\xi))}{\varphi(y(\xi))}. \tag{2.15}$$

Hence,

$$\begin{aligned}x^{\Delta}(\xi) &= \frac{\left[r(\xi)\varphi(y^{\Delta}(\xi))\right]^{\Delta}}{\varphi(y(\xi))} + \left[r(\xi)\varphi(y^{\Delta}(\xi))\right]^{\sigma}\left(\frac{1}{\varphi(y(\xi))}\right)^{\Delta}\\ &= -\frac{\varphi(y(g(\xi)))}{\varphi(y(\xi))}q(\xi) - \frac{[\varphi(y(\xi))]^{\Delta}}{\varphi(y(\xi))}x(\sigma(\xi))\\ &\leq -\left(\frac{R(g(\xi))}{R(\xi)}\right)^{\gamma\sqrt[\gamma]{\beta_n}} q(\xi) - \frac{[\varphi(y(\xi))]^{\Delta}}{\varphi(y(\xi))}x(\sigma(\xi)).\end{aligned} \tag{2.16}$$

(I) $0 < \gamma \leq 1$. The result of applying Pötzsche chain rule is

$$\begin{aligned}\frac{[\varphi(y(\xi))]^{\Delta}}{\varphi(y(\xi))} &\geq \gamma\left(\frac{y(\xi)}{y(\sigma(\xi))}\right)^{1-\gamma}\frac{y^{\Delta}(\xi)}{y(\xi)}\\ &= \gamma\left(\frac{y(\xi)}{y(\sigma(\xi))}\right)^{1-\gamma}\left(\frac{x(\xi)}{r(\xi)}\right)^{1/\gamma}.\end{aligned} \tag{2.17}$$

By using the facts that $y(\xi) > 0$ and $\left[r(\xi)\varphi(y^{\Delta}(\xi))\right]^{\Delta} < 0$, we get

$$y(\xi) \geq r^{1/\gamma}(\xi) y^{\Delta}(\xi) R(\xi),$$

and so

$$\frac{y(\xi)}{y(\sigma(\xi))} = \frac{y(\xi)}{y(\xi) + \mu(\xi) y^{\Delta}(\xi)} \geq \frac{1}{1 + \frac{\mu(\xi)}{r^{1/\gamma}(\xi) R(\xi)}} = \frac{R(\xi)}{R(\xi) + \frac{\mu(\xi)}{r^{1/\gamma}(\xi)}} = \frac{R(\xi)}{R(\sigma(\xi))}.$$

Hence, (2.17) becomes

$$\frac{[\varphi(y(\xi))]^{\Delta}}{\varphi(y(\xi))} \geq \gamma \left(\frac{R(\xi)}{R(\sigma(\xi))}\right)^{1-\gamma} \left(\frac{x(\xi)}{r(\xi)}\right)^{1/\gamma}. \tag{2.18}$$

Using (2.18) in (2.16) , we get for $\xi \in [\xi_1, \infty)_{\mathbb{T}}$,

$$x^{\Delta}(\xi) \leq -\left(\frac{R(g(\xi))}{R(\xi)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(\xi) - \frac{\gamma}{r^{1/\gamma}(\xi)} \left(\frac{R(\xi)}{R(\sigma(\xi))}\right)^{1-\gamma} x^{1/\gamma}(\xi) x(\sigma(\xi)), \tag{2.19}$$

which yields that $x^{\Delta} < 0$. By integrating (2.19) from ξ to v, we infer that

$$\begin{aligned} x(v) - x(\xi) \leq & -\int_{\xi}^{v} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t) \Delta t \\ & -\int_{\xi}^{v} \frac{\gamma}{r^{1/\gamma}(t)} \left(\frac{R(t)}{R(\sigma(t))}\right)^{1-\gamma} x^{1/\gamma}(t) x(\sigma(t)) \Delta t. \end{aligned}$$

Considering that $x > 0$ and passing the limit as $v \to \infty$, we find that

$$\begin{aligned} -x(\xi) \leq & -\int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t) \Delta t \\ & -\int_{\xi}^{\infty} \frac{\gamma}{r^{1/\gamma}(t)} \left(\frac{R(t)}{R(\sigma(t))}\right)^{1-\gamma} x^{1/\gamma}(t) x(\sigma(t)) \Delta t; \end{aligned}$$

that is,

$$\begin{aligned} x(\xi) \geq & \int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t) \Delta t \\ & + \int_{\xi}^{\infty} \frac{\gamma}{r^{1/\gamma}(t)} \left(\frac{R(t)}{R(\sigma(t))}\right)^{1-\gamma} x^{1/\gamma}(t) x(\sigma(t)) \Delta t. \end{aligned} \tag{2.20}$$

Now, for any $\varepsilon \in (0, 1)$, there exists a $\xi_2 \in [\xi_1, \infty)_{\mathbb{T}}$ such that for $\xi \in [\xi_2, \infty)_{\mathbb{T}}$,

$$\int_{\xi}^{\infty} \left(\frac{R(g(t))}{R(t)}\right)^{\gamma \sqrt[\gamma]{\beta_n}} q(t) \Delta t \geq \frac{\varepsilon \beta_{n+1}}{R^{\gamma}(\xi)}, \frac{R(\xi)}{R(\sigma(\xi))} \geq \varepsilon l, \quad \text{and } R^{\gamma}(\xi) x(\xi) \geq \varepsilon R_*, \tag{2.21}$$

where

$$R_* := \liminf_{\xi\to\infty} R^\gamma(\xi)\, x(\xi),\ \ 0 \le r_* \le 1.$$

In view of (2.20) and (2.21) , we have

$$\begin{aligned} x(\xi) &\ge \frac{\varepsilon\beta_{n+1}}{R^\gamma(\xi)} + (\varepsilon R_*)^{1+1/\gamma} \int_\xi^\infty \frac{\gamma}{r^{1/\gamma}(t) R^\gamma(t)\, R(\sigma(t))} \Delta t \\ &\ge \frac{\varepsilon\beta_{n+1}}{R^\gamma(\xi)} + \varepsilon^{2-\gamma+1/\gamma}\, l^{1-\gamma}\, R_*^{1+1/\gamma} \int_\xi^\infty \frac{\gamma}{r^{1/\gamma}(t) R(t)\, R^\gamma(\sigma(t))} \Delta t \\ &\ge \frac{\varepsilon\beta_{n+1}}{R^\gamma(\xi)} + \varepsilon^{2-\gamma+1/\gamma}\, l^{1-\gamma}\, R_*^{1+1/\gamma} \int_\xi^\infty \left(\frac{-1}{R^\gamma(t)}\right)^\Delta \Delta t \\ &= \frac{\varepsilon\beta_{n+1}}{R^\gamma(\xi)} + \varepsilon^{2-\gamma+1/\gamma}\, l^{1-\gamma}\, R_*^{1+1/\gamma} \frac{1}{R^\gamma(\xi)}, \end{aligned} \tag{2.22}$$

since by the quotient and Pötzsche chain rule

$$\left(\frac{-1}{R^\gamma(t)}\right)^\Delta = \frac{(R^\gamma(t))^\Delta}{R^\gamma(t)\, R^\gamma(\sigma(t))} \le \frac{\gamma}{r^{1/\gamma}(t) R(t)\, R^\gamma(\sigma(t))}.$$

By multiplying (2.22) by $R^\gamma(\xi)$, we conclude that

$$R^\gamma(\xi)\, x(\xi) \ge \varepsilon\beta_{n+1} + \varepsilon^{2-\gamma+1/\gamma}\, l^{1-\gamma}\, R_*^{1+1/\gamma}.$$

Taking the lim inf as $\xi \to \infty$, we get

$$R_* \ge \varepsilon\beta_{n+1} + \varepsilon^{2-\gamma+1/\gamma}\, l^{1-\gamma}\, R_*^{1+1/\gamma}.$$

Since $0 < \varepsilon < 1$ is arbitrary, we find that

$$\beta_{n+1} \le R_* - l^{1-\gamma} R_*^{1+1/\gamma}.$$

Let

$$A = 1,\ B = l^{1-\gamma},\ \text{and } u = R_*.$$

Using the inequality

$$Au - Bu^{1+1/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{A^{\gamma+1}}{B^\gamma},\ B > 0, \tag{2.23}$$

we see that

$$\beta_{n+1} \le \frac{\gamma^\gamma}{l^{\gamma(1-\gamma)}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.14) with $0 < \gamma \le 1$.

(II) $\gamma \ge 1$. Again, applying Pötzsche chain rule, we have

$$\frac{[\varphi(y(\xi))]^\Delta}{\varphi(y(\xi))} \ge \gamma \frac{y^\Delta(\xi)}{y(\xi)} = \gamma \left(\frac{x(\xi)}{r(\xi)}\right)^{1/\gamma}.$$

Hence, (2.16) becomes

$$x^{\Delta}(\xi) \leq -q(\xi)\left(\frac{R(g(\xi))}{R(\xi)}\right)^{\gamma\sqrt[\gamma]{\beta_n}} - \frac{\gamma}{r^{1/\gamma}(\xi)}x^{1/\gamma}(\xi)x(\sigma(\xi)). \tag{2.24}$$

By integrating (2.24) from ξ to v and letting $v \to \infty$, we deduce that

$$x(\xi) \geq \int_{\xi}^{\infty} q(t)\left(\frac{R(g(t))}{R(t)}\right)^{\gamma\sqrt[\gamma]{\beta_n}}\Delta t + \int_{\xi}^{\infty}\frac{\gamma}{r^{1/\gamma}(t)}x^{1/\gamma}(t)x(\sigma(t))\,\Delta t. \tag{2.25}$$

Using (2.21) in (2.25) , we get

$$\begin{aligned} x(\xi) &\geq \frac{\varepsilon\beta_{n+1}}{R^{\gamma}(\xi)} + (\varepsilon R_*)^{1+1/\gamma}\int_{\xi}^{\infty}\frac{\gamma}{r^{1/\gamma}(t)R(t)R^{\gamma}(\sigma(t))}\Delta t \\ &\geq \frac{\varepsilon\beta_{n+1}}{R^{\gamma}(\xi)} + \varepsilon^{\gamma+1/\gamma}l^{\gamma-1}R_*^{1+1/\gamma}\int_{\xi}^{\infty}\frac{\gamma}{r^{1/\gamma}(t)R^{\gamma}(t)R(\sigma(t))}\Delta t \\ &\geq \frac{\varepsilon\beta_{n+1}}{R^{\gamma}(\xi)} + \varepsilon^{\gamma+1/\gamma}l^{\gamma-1}R_*^{1+1/\gamma}\int_{\xi}^{\infty}\left(\frac{-1}{R^{\gamma}(t)}\right)^{\Delta}\Delta t \\ &= \frac{\varepsilon\beta_{n+1}}{R^{\gamma}(\xi)} + \varepsilon^{\gamma+1/\gamma}l^{\gamma-1}R_*^{1+1/\gamma}\frac{1}{R^{\gamma}(\xi)}, \end{aligned}$$

since by the quotient and Pötzsche chain rule

$$\left(\frac{-1}{R^{\gamma}(t)}\right)^{\Delta} = \frac{(R^{\gamma}(t))^{\Delta}}{R^{\gamma}(t)R^{\gamma}(\sigma(t))} \leq \frac{\gamma}{r^{1/\gamma}(t)R^{\gamma}(t)R(\sigma(t))}.$$

The rest of the proof is similar to Part (I) and is therefore omitted.

□

Example 2.1. Consider the second-order half-linear advanced dynamic equation

$$\left(\frac{1}{9\xi^2}\left(y^{\Delta}(\xi)\right)^2\operatorname{sgn}y^{\Delta}(\xi)\right)^{\Delta} + \frac{1}{4\xi^5}y^2(2\xi)\operatorname{sgn}y(2\xi) = 0,\ \xi \in [1,\infty), \tag{2.26}$$

where $r(\xi) = \frac{1}{9\xi^2}$, $\gamma = 2$, $q(\xi) = \frac{1}{4\xi^5}$, and $g(\xi) = 2\xi$. Now

$$\int_{\xi_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)} = 3\int_{\xi_0}^{\infty} t\,dt = \infty,$$

and

$$\beta_{n+1} = \frac{9}{16}\liminf_{\xi\to\infty}\left(\xi^2-1\right)^2\int_{\xi}^{\infty}\frac{1}{t^5}\left(\frac{4t^2-1}{t^2-1}\right)^{2\sqrt{\beta_n}}dt,\ \frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}} = 0.148148.$$

Therefore,

$$\beta_1 = \frac{9}{16}\liminf_{\xi\to\infty}\left(\xi^2-1\right)^2\int_{\xi}^{\infty}\frac{1}{t^5}\left(\frac{4t^2-1}{t^2-1}\right)^{2\sqrt{\beta_0}}dt = 0.140625 < \frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}}$$

and

$$\beta_2 = \frac{9}{16}\liminf_{\xi\to\infty}\left(\xi^2-1\right)^2\int_{\xi}^{\infty}\frac{1}{t^5}\left(\frac{4t^2-1}{t^2-1}\right)^{2\sqrt{\beta_1}}dt = 0.397747 > \frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}}.$$

Then, by Theorem 2.1, Equation (2.26) is oscillatory. Clearly, Theorem 1.1 is not applicable to Equation (2.26) since

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}q(t)\,\Delta t\right\}<\frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}}.$$

3 | DISCUSSION AND CONCLUSIONS

1) In this paper, Hille-type iterative criteria are given including a role for $g(\xi)$ that can be applied to Equation (1.1) and are valid for a variety of time scales, such as $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, etc. (see [4]). Moreover, Theorem 2.1 improves Theorem 1.1 because of

$$\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}\left(\frac{R(g(t))}{R(t)}\right)^{\gamma\sqrt[\gamma]{\beta_n}}q(t)\,\Delta t\right\}\geq\liminf_{\xi\to\infty}\left\{R^{\gamma}(\xi)\int_{\xi}^{\infty}q(t)\,\Delta t\right\}.$$

2) The result of Theorem 2.1 not only enhances but also expands Theorem 1.1. In particular, if (2.14) is satisfied with $n\geq 1$ and

$$0<\beta_{i+1}\leq\frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}},\ i=0,1,\ \dots\ ,n-1\ \text{ and }\ \beta_{n+1}>\frac{\gamma^{\gamma}}{l^{\gamma|\gamma-1|}(\gamma+1)^{\gamma+1}},$$

then by Theorem 2.1, we know that Equation (1.1) is oscillatory, but Theorem 1.1 cannot be applied.

3) It would be intriguing to find a Hille-type condition for Equation (1.1) in the case when $\int_{\xi_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}<\infty$.

AUTHOR CONTRIBUTIONS

Taher S. Hassan: Writing—original draft; writing—review and editing; supervision; formal analysis; validation; investigation; and methodology. **Clemente Cesarano:** Writing—review and editing; resources; supervision; and formal analysis. **Mouataz Billah Mesmouli:** Writing—review and editing; formal analysis; resources; and data curation. **Hasan Nihal Zaidi:** Writing—review and editing; resources; investigation; and data curation. **Ismoil Odinaev:** Writing—review and editing; resources; and data curation.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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REFERENCES

1. S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.
2. V. Kac and P. Chueng, *Quantum calculus*, Universitext, 2002.
3. R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), 1–26.
4. M. Bohner and A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, Boston, 2001.
5. M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.
6. L. E. Elsgolts and S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*, Elsevier, 1973.
7. I. Gyori and G. Ladas, *Oscillation theory of delay differential equations with applications*, Clarendon Press, Oxford, 1991.
8. S. H. Saker, *Oscillation theory of dynamic equations on time scales: second and third orders lap lambert*, Academic Publishing, 2010.
9. M. B. Mesmouli and T. S. Hassan, *On the positive solutions for IBVP of conformable differential equations*, AIMS Math. **8** (2023), no. 10, 24740–24750.
10. J. Ohriska, *Oscillation of second order delay and ordinary differential equations*, Czech. Math. J. **34** (1984), 107–112.
11. R. P. Agarwal, S. L. Shieh, and C. C. Yeh, *Oscillation criteria for second-order retarded differential equations*, Math. Comput. Modelling **26** (1997), 1–11.
12. S. Sun, Z. Han, P. Zhao, and C. Zhang, *Oscillation for a class of second-order Emden-Fowler delay dynamic equations on time scales*, Adv. Differ. Equ-Ny. **2010**, 642356.

13. B. Baculikova, *Oscillation of second-order nonlinear noncanonical differential equations with deviating argument*, Appl. Math. Lett. **91** (2019), 68–75.
14. O. Bazighifan and E. M. El-Nabulsi, *Different techniques for studying oscillatory behavior of solution of differential equations*, Rocky Mountain J. Math. **51** (2021), 77–86. https://projecteuclid.org/euclid.rmjm/1596037174
15. J. Džurina and I. A. Jadlovská, *Sharp oscillation result for second-order half-linear noncanonical delay differential equations*, J. Qual. Theo. **46** (2020), 1–14.
16. J. Džurina and I. Jadlovská, *A note on oscillation of second-order delay differential equations*, Appl. Math. Lett. **69** (2017), 126–132.
17. L. Erbe, T. S. Hassan, and A. Peterson, *Oscillation criteria for second order sublinear dynamic equations with damping term*, J. Difference Equ. Appl. **17** (2011), 505–523.
18. L. Erbe and T. S. Hassan, *New oscillation criteria for second order sublinear dynamic equations*, Dynam. Syst. Appl. **22** (2013), 49–63.
19. S. R. Grace, M. Bohner, and R. P. Agarwal, *On the oscillation of second-order half-linear dynamic equations*, J. Differ. Equ. Appl. **15** (2009), 451–460.
20. Y. R. Zhu, Z. X. Mao, S. P. Liu, and J. F. Tian, *Oscillation criteria of second-order dynamic equations on time scales*, Mathematics **9** (2021), 1867.
21. Q. Zhang, L. Gao, and L. Wang, *Oscillation of second-order nonlinear delay dynamic equations on time scales*, Comput. Math. Appl. **61** (2011), 2342–2348.
22. I. Jadlovská, *Iterative oscillation results for second-order differential equations with advanced argument*, Electron. J. Differ. Equation **2017** (2017), no. 162, 1–11.
23. M. Bohner, K. S. Vidhyaa, and E. Thandapani, *Oscillation of noncanonical second-order advanced differential equations via canonical transform*, Construct. Math. Anal. **5** (2022), no. 1, 7–13.
24. G. E. Chatzarakis, J. Džurina, and I. Jadlovská, *New oscillation criteria for second-order half-linear advanced differential equations*, Appl. Math. Comput. **347** (2019), 404–416.
25. G. E. Chatzarakis, O. Moaaz, T. Li, and B. Qaraad, *Some oscillation theorems for nonlinear second-order differential equations with an advanced argument*, Adv. Differ. Equ. **2020** (2020), 160.
26. T. Li and Y. V. Rogovchenko, *Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations*, Monatsh. Math. **184** (2017), no. 3, 489–500.
27. T. Li and Y. V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett. **105** (2020), 106293.
28. S. Frassu and G. Viglialoro, *Boundedness in a chemotaxis system with consumed chemoattractant and produced chemorepellent*, Nonlinear Anal. **213** (2021), 112505.
29. T. Li and G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime*, Differ. Integral Equat. **34** (2021), no. 5-6, 315–336.
30. A. Columbu, S. Frassu, and G. Viglialoro, *Properties of given and detected unbounded solutions to a class of chemotaxis models*, Stud. Appl. Math. **151** (2023), 1349–1379.
31. T. Li, S. Frassu, and G. Viglialoro, *Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption*, Z. Angew. Math. Phys. **74** (2023), no. 3, 109.
32. T. Ren, S. Li, X. Zhang, and L. Liu, *Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes*, Bound. Value Probl. **2017** (2017), 118.
33. J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, *Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation*, Math. Modell. Anal. **23** (2018), 611–626.
34. X. Zhang, P. Xu, Y. Wu, and B. Wiwatanapataphe, *The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model*, Nonlin. Anal.: Modell. Control **27** (2022), no. 3, 428–444.
35. X. Zhang, P. Chen, H. Tian, and Y. Wu, *The iterative properties for positive solutions of a tempered fractional equation*, Fractal Fract. **7** (2023), 761.
36. R. P. Agarwal, M. Bohner, and T. Li, *Oscillatory behavior of second-order half-linear damped dynamic equations*, Appl. Math. Comput. **254** (2015), 408–418.
37. M. Bohner and T. Li, *Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient*, Appl. Math. Lett. **37** (2014), 72–76.
38. M. Bohner and T. Li, *Kamenev-type criteria for nonlinear damped dynamic equations*, Sci. China Math. **58** (2015), no. 7, 1445–1452.
39. T. Li, N. Pintus, and G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys. **70** (2019), no. 3, 86.
40. C. Zhang, R. P. Agarwal, M. Bohner, and T. Li, *Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 2, 761–778.
41. R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, *Oscillation criteria for second-order dynamic equations on time scales*, Appl. Math. Lett. **31** (2014), 34–40.
42. J. Řezníčková, *Hille-nehari type oscillation and nonoscillation criteria for linear and half-linear differential equations*, MATEC Web Conf. **292** (2019), 1061.
43. B. Baculikova, *Oscillation and asymptotic properties of second order half-linear differential equations with mixed deviating arguments*, Mathematics **9** (2021), 2552.

44. G. V. Demidenko and I. I. Matveeva, *Asymptotic stability of solutions to a class of second-order delay differential equations*, Mathematics **9** (2021), 1847.
45. E. Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234–252.
46. L. Erbe, *Oscillation criteria for second order nonlinear delay equations*, Canad. Math. Bull. **16** (1973), 49–56.
47. R. P. Agarval, S. R. Grace, and D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
48. O. Došlý and P. Řehák, *Half-linear differential equations*, North Holland Mathematics Studies, Vol. **202**, Elsevier, Amsterdam, 2005.
49. T. Kusano and M. Naito, *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. **33** (1981), no. 3, 509–533.
50. J. Džurina, *Oscillation of second order advanced differential equations*, Electron. J. Qual. Theo. **20** (2018), 1–9.
51. T. S. Hassan, Q. Kong, and B. M. El-Matary, *Oscillation criteria for advanced half-linear differential equations of second order*, Mathematics **11** (2023), 1385.
52. L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, *Oscillation criteria for half-linear delay dynamic equations on time scales*, Nonlinear Dynam. Sys. Th. **9** (2009), 51–68.
53. L. Erbe, T. S. Hassan, A. Peterson, and S. H. Saker, *Oscillation criteria for sublinear half-linear delay dynamic equations on time scales*, Int. J. Differ. Equ. **3** (2008), 227–245.
54. M. Bohner, T. S. Hassan, and T. Li, *Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments*, Indag. Math. **29** (2018), 548–560.
55. B. Karpuz, *Hille–Nehari theorems for dynamic equations with a time scale independent critical constant*, Appl. Math. Comput. **346** (2019), 336–351.
56. T. S. Hassan, R. P. Agarwal, and W. Mohammed, *Oscillation criteria for third-order functional half-linear dynamic equations*, Adv. Difference Equ. **2017** (2017), 111.
57. T. S. Hassan, Y. Sun, and A. Abdel Menaem, *Improved oscillation results for functional nonlinear dynamic equations of second order*, Mathematics **8** (2020), 1897.
58. T. S. Hassan, R. A. El-nabulsi, and A. Abdel Menaem, *Amended criteria of oscillation for nonlinear functional dynamic equations of second-order*, Mathematics **9** (2021), 1191.
59. T. S. Hassan, C. Cesarano, R. A. El-nabulsi, and W. Anukool, *Improved Hille-type oscillation criteria for second-order quasilinear dynamic equations*, Mathematics **10** (2022), 3675.
60. X. Yang, *A note on oscillation and nonoscillation for second-order linear differential equation*, J. Math. Anal. Appl. **238** (1999), 587–590.
61. S. Fišnarová and Z. Pátíková, *Hille–Nehari type criteria and conditionally oscillatory half-linear differential equations*, Electron. J. Qual. Theo. **2019** (2019), 1–22.
62. P. Řehák, *New results on critical oscillation constants depending on a graininess*, Dynam. Syst. Appl. **19** (2010), 271–288.
63. P. Řehák, *A critical oscillation constant as a variable of time scales for half-linear dynamic equations*, Math. Slovaca **60** (2010), 237–256.
64. T. S. Hassan, M. Bohner, I. L. Florentina, A. Abdel Menaem, and M. B. Mesmouli, *New Criteria of oscillation for linear Sturm–Liouville delay noncanonical dynamic equations*, Mathematics **11** (2023), 4850.
65. W. B. Fite, *Concerning the zeros of the solutions of certain differential equations*, Trans. Amer. Math. Soc. **19** (1918), 341–352.
66. L. Erbe, T. S. Hassan, and A. Peterson, *Oscillation criteria for nonlinear damped dynamic equations on time scales*, Appl. Math. Comput. **203** (2008), 343–357.

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